The Radical of Binary Dimensional Dual Hyperovals

Ulrich Dempwolff
Department of Mathematics,
University of Kaiserslautern,
Kaiserslautern, Germany

Yves Edel
Department of Mathematics:
Analysis, Logic and Discrete Mathematics,
Ghent University,
Ghent, Belgium

Abstract

Let $\mathcal{S}$ be a dimensional dual hyperoval of rank n over $\mathbb{F}_2$. We introduce and study the radical $P(\mathcal{S})$, which is a subspace of the ambient space $U(\mathcal{S})$ of $\mathcal{S}$ invariant under the automorphism group of $\mathcal{S}$. For the vast majority of the known dimensional dual hyperovals we have $P(\mathcal{S}) = U(\mathcal{S})$. Interesting is the case of proper radicals, i.e. $P(\mathcal{S}) \neq U(\mathcal{S})$. Starting point of our investigations is a result of the second author [10, Thm. 1], [7, Thm. 3.6] (Theorem 1.2 below) which characterizes alternating dual hyperovals by the property that $\mathcal{S}$ splits over $P(\mathcal{S})$. This Theorem is extended by Theorem 1.3 where we characterize dimensional dual hyperovals $\mathcal{S}$ with $\dim U(\mathcal{S}) - \dim P(\mathcal{S}) = \mathrm{rank}(\mathcal{S}) - 1$. Moreover we will show (Theorem 4.6) that a proper radical implies that this dimensional dual hyperoval is a disjoint union of subDHOs of smaller rank. The notion of "disjoint union of subDHOs" has been introduced by Yoshiara [17]. Some theory on dimensional dual hyperovals with proper radicals is developed. Our paper also provides some computational results on dual hyperovals of small rank with a proper radical. These calculations indicate — though dual hyperovals with a proper radical seem to be scarce — that the number of these hyperovals is steadily growing as function of the rank.

Keywords: dimensional dual hyperoval
MSC: 51A45, 05B25

1 Introduction

Let $n \geq 2$. A set $\mathcal{S}$ of size $|\mathcal{S}| = (q^n - 1)/(q - 1) + 1$ of n-dimensional subspaces of a finite $\mathbb{F}_q$-vector space is called a *dual hyperoval of rank n* (we will use in the sequel the abbreviation DHO), if for every $X \in \mathcal{S}$ and every 1-space $P \subseteq X$ there exists precisely one $X' \in \mathcal{S} - \{X\}$ such that $X \cap X' = P$. The space $U(\mathcal{S}) = \langle X \mid X \in \mathcal{S} \rangle$ is called the *ambient space* of the DHO. Often, a DHO of rank n is viewed projectively and called a $(n-1)$-*dimensional dual hyperoval*. In this paper, we prefer to take the point of view of vector spaces, and hence all dimensions will be vector space dimensions. For basic definitions and background information on DHOs we refer to Yoshiara [16].

Definition 1.1. Let $\mathcal{S}$ be a DHO over $\mathbb{F}_2$ and $X, X' \in \mathcal{S}$. Define $X \curlywedge X'$ as the nontrivial vector in $X \cap X'$ if $X \neq X'$ and if $X = X'$ we set $X \curlywedge X' = 0$. For three members $X_1, X_2, X_3 \in \mathcal{S}$ define

$$u(X_1, X_2, X_3) = X_1 \curlywedge X_2 + X_1 \curlywedge X_3 + X_2 \curlywedge X_3$$

and define in the ambient space the subspace

$$P(\mathcal{S}) = \langle u(X_1, X_2, X_3) \mid X_1, X_2, X_3 \in \mathcal{S} \rangle$$

and call $P(\mathcal{S})$ the *radical of* $\mathcal{S}$. We call the radical *proper* if $P(\mathcal{S}) \neq U(\mathcal{S})$.

A DHO $\mathcal{S}$ is of *split type* or *splits over* Y if there exists a subspace $Y \subseteq U(\mathcal{S})$ with

$$U(\mathcal{S}) = X \oplus Y$$

for all $X \in \mathcal{S}$. Edel [10, Thm. 1], [7, Thm. 3.6] gives the following geometric characterization of alternating DHOs:

Theorem 1.2. *Let $\mathcal{S}$ be a DHO over $\mathbb{F}_2$. Equivalent are:*

(a) $\mathcal{S}$ splits over $P(\mathcal{S})$.

(b) $\mathcal{S}$ is an alternating DHO.

An inspection of known DHOs over $\mathbb{F}_2$ shows, that given a DHO $\mathcal{S}$ over $\mathbb{F}_2$, one can expect in the majority of cases that the radical is not proper. On the other hand inspecting the known DHOs of rank 4 (see [1]) one observes that the difference $\dim U(\mathcal{S}) - \dim P(\mathcal{S})$ takes all values in the range between 0 and 4 (by Lemma 2.1 this difference is bounded from above by the rank of a DHO). Also $P(\mathcal{S})$ is (obviously) invariant under the automorphism group of $\mathcal{S}$. This motivates an investigation of the radical of a DHO.

In the next Section we prove basic properties of the radical. Quotients of the Huybrechts DHO and the Buratti-Del Fra DHO provide examples with a proper radical as we see in Section 3. In Section 4 we define substructures of DHOs which are in the most natural way DHOs of lower rank — so called subDHOs — embedded in the given DHO. It turns out that the existence of subDHOs has to do with the property that the radical is proper. Furthermore we see that this property is the basis for Yoshiara's work on disjoint unions of subDHOs [17]. In Section 5 Theorem 5.1 describes the radical of bilinear DHOs. Some consequences of this Theorem are discussed. In Section 6 we shall prove the following:

Theorem 1.3. *Let $n \geq 4$ and $\mathcal{S}$ be a DHO of rank n over $\mathbb{F}_2$ such that $\dim U(\mathcal{S}) - \dim P(\mathcal{S}) = n-1$. Then $\mathcal{S}$ is a quotient of $\mathcal{H}_n$ or $\mathcal{D}_n$.*

Here $\mathcal{H}_n$ stands for the Huybrechts DHO of rank n and $\mathcal{D}_n$ stands for the Buratti-Del Fra DHO of rank n. It is known that alternating DHOs are quotients of Huybrechts DHOs (see [5, Lem. 2.2]). In Section 7 we present some sporadic examples with proper radical not covered by the general examples of Section 3.

2 Properties of the radical

The radical of a DHO is a supplement of a DHO, namely;

Lemma 2.1. *$U(\mathcal{S}) = X + P(\mathcal{S})$ for every $X \in \mathcal{S}$. In particular $\mathcal{S}$ splits over $P(\mathcal{S})$ if $\dim U(\mathcal{S}) - \dim P(\mathcal{S}) = n$.*

Proof. As $X \curlywedge X_1 \in X$ we see $X_2 \curlywedge X_3 = u(X, X_2, X_3) + X \curlywedge X_2 + X \curlywedge X_3 \in P(\mathcal{S}) + X$. So $U(\mathcal{S}) = \langle X'' \curlywedge X' \mid X'' \neq X' \rangle \subseteq X + P(\mathcal{S})$. □

As an immediate corollary we obtain the following extension of Theorem 1.2:

Corollary 2.2. *Let $\mathcal{S}$ be a DHO over $\mathbb{F}_2$ of rank n. Equivalent are:*

(a) *$\mathcal{S}$ splits over $P(\mathcal{S})$.*

(b) *$\mathcal{S}$ is an alternating DHO.*

(c) $\dim U(\mathcal{S}) = \dim P(\mathcal{S}) + n$.

Lemma 2.1 raises an immediate question. Suppose that the DHO is of split type. Does the radical contain a complement? We can not answer this question but at least for bilinear DHOs this question has a positive answer.

Let X, Y be finite dimensional $\mathbb{F}_2$-spaces, $\beta : X \to \mathrm{Hom}(X, Y)$ a monomorphism. For $e \in X$ define by $X(e) = \{(x, x\beta(e)) \mid x \in X\}$ a subspace in $U = X \oplus Y$. If $\mathcal{S}_\beta = \{X(e) \mid x \in X\}$ is a DHO in U we call $\mathcal{S}_\beta$ *bilinear*. A bilinear DHO is *alternating* if $e\beta(e) = 0$ for all $e \in X$. The mappings $\tau_e \in \mathrm{GL}(U)$ defined by $(x, y)\tau_e = (x, y + x\beta(e))$ are automorphisms of this DHO and they form the standard translation group $T = T_\beta$ of $\mathcal{S}_\beta$ (for more basis information on bilinear DHOs see [7]). Note, that $Y = C_U(T) = [U, T]$ and that $\mathcal{S}_\beta$ splits over Y where $C_U(T) = \{u \in U \mid u\tau = u,\, \tau \in T\}$ is the *centralizer of T in U* and $[U, T] = \langle u\tau + u \mid u \in U,\, \tau \in T\rangle$ is the *commutator of U and T*.

For the computation of the radical we define the kernel function $\kappa = \kappa_\beta : X \to X$ by $\kappa(0) = 0$ and for $0 \neq e \in X$ denote by $\kappa(e)$ the generator of $\ker \beta(e)$. Note, that κ is a bijection on X by the definition of a DHO. One observes:

Lemma 2.3. *Let X, Y be $\mathbb{F}_2$-spaces, $\beta : X \to \mathrm{Hom}(X, Y)$ be a monomorphism which defines a bilinear DHO $\mathcal{S} = \mathcal{S}_\beta$ with ambient space $U = X \oplus Y$. Set $X_\kappa = \langle \kappa(e) + \kappa(f) + \kappa(e + f) \mid 0 \neq e, f \in X\rangle$. Then $P(\mathcal{S}) \subseteq X_\kappa \oplus Y$ and $P(\mathcal{S})$ contains a T-invariant complement (T the standard translation group). In particular $P(\mathcal{S})$ is proper iff X_κ is a proper subspace of X.*

Theorem 5.1 below improves this lemma significantly.

Proof. As $\kappa(e+f)\beta(e) = \kappa(e+f)\beta(f)$

$$X(e) \curlywedge X(f) = (\kappa(e+f), \kappa(e+f)\beta(e)).$$

So

$$u(X(0), X(e), X(f)) = (\kappa(e) + \kappa(f) + \kappa(e+f), \kappa(e+f)\beta(e)) \in X_\kappa \oplus Y.$$

Also $u(X(h), X(e+h), X(f+h)) = u(X(0), X(e), X(f))\tau_h \equiv u(X(0), X(e), X(f))$ (mod Y) showing $P(\mathcal{S}) \subseteq X_\kappa \oplus Y$. Set $P_Y = P(\mathcal{S}) \cap Y$ and let $\{y_i + P_Y \mid 1 \leq i \leq r\}$ be a basis of Y/P_Y. By Lemma 2.1 there exist $x_i \in X$, $z_i \in P_Y$ such that $v_i = x_i + y_i + z_i \in P(\mathcal{S})$. Then $x_i \in X_\kappa$. Set $K = \langle v_i, P_Y \mid 1 \leq i \leq r\rangle$. Then K is T-invariant since $[K,T] \subseteq [X_\kappa \oplus Y, T] \subseteq P(\mathcal{S}) \cap Y = P_Y \subseteq K$. Clearly, $\dim K = \dim Y$ and $X \cap K = 0$. Since K is T-invariant $\mathcal{S}$ splits over K. As $Y = [X,T]$ Lemma 2.1 implies the last assertion. □

Let $\mathcal{S}$, $\mathcal{S}'$ be DHOs of rank n. A linear mapping $\phi : U(\mathcal{S}) \to U(\mathcal{S}')$ is a *covering map* if $\mathcal{S}' = \{X\phi \mid X \in \mathcal{S}\}$. One says that $\mathcal{S}$ is a *cover* of $\mathcal{S}'$ and that $\mathcal{S}'$ is a *quotient* of $\mathcal{S}$. In this situation $\mathcal{S}' \simeq \mathcal{S}/W$ where $W = \ker\phi$ and $\mathcal{S}/W = \{(X+W)/W \mid X \in \mathcal{S}\}$. A cover is *proper* if ϕ is not an isomorphism. DHOs which do not have a proper cover are called *simply connected*. For every DHO there exists a unique simply connected cover, the *universal cover* (see [16, Def. 2.10] and [4]). The behavior of the radical operator under homomorphisms is described by:

Proposition 2.4. *Let $\mathcal{S}$ be a DHO over $\mathbb{F}_2$ and $W \subseteq U(\mathcal{S})$ a subspace defining the quotient $\mathcal{S}/W$ in $U(\mathcal{S}/W) = U(\mathcal{S})/W$. Then $P(\mathcal{S}/W) = (P(\mathcal{S}) + W)/W$.*

Proof. As $\mathcal{S}/W$ is a DHO we have $\dim((X+W) \cap (X'+W)) = \dim W + 1$ for $X, X' \in \mathcal{S}$ distinct. The space $(X \cap X') \oplus W$ lies in $(X+W) \cap (X'+W)$ and has the same dimension. Hence $(X+W)/W \cap (X'+W)/W = ((X \cap X') + W)/W$. The assertion follows. □

3 Examples with a proper radical

In this Section we consider the radical of the Buratti-Del Fra DHOs $\mathcal{D}_n$ and quotients of Huybrechts DHOs $\mathcal{H}_n$ with a proper radical.

3.1 Buratti-Del Fra DHOs

We recall a construction of the Buratti-Del Fra DHOs. Let $X = X_n = \langle e_0, e_1, \ldots, e_{n-1}\rangle$ be an n-dimensional $\mathbb{F}_2$-space and denote by $S^2(X)$ the second component of the symmetric algebra over X (which is generated by the vectors $e_i \cdot e_j$, $i \leq j$). Set $R_n = \langle e_0^2, e_0 \cdot e_i + e_i^2 \mid 0 < i < n\rangle$ and $Y = Y_n = S^2(X)/R_n$. For $u, v \in V_n$ we will denote by $\overline{u \cdot v}$ the homomorphic image of $u \cdot v \in S^2(X)$ in Y. Define in

$U_n = X_n \oplus Y_n$ the n-dimensional Buratti-Del Fra DHO by $\mathcal{D}_n = \{X(e) \mid e \in X\}$, where $X(e) = \{(x, \overline{x \cdot e}) \mid x \in X\}$. Note, that the map

$$X \times X \ni (x, e) \mapsto \overline{x \cdot e} \in Y$$

is bilinear and symmetric and that $\overline{x \cdot x} = \overline{x \cdot e_0}$.

Lemma 3.1. *Let $\mathcal{D}_n$ be the Buratti-Del Fra of rank $n \geq 3$.*

(a) Let $n = 3$. Then $P(\mathcal{D}_n) = \langle e_0 \rangle \oplus \langle \overline{e_0 \cdot e_1}, \overline{e_0 \cdot e_2} \rangle$.

(b) Let $n > 3$. Then $P(\mathcal{D}_n) = \langle e_0 \rangle \oplus Y$.

Proof. Set $x\beta(e) = \overline{x \cdot e}$ for $x, e \in X$ and $\kappa = \kappa_\beta$. Since $\overline{x \cdot x} = \overline{x \cdot e_0}$, i.e. $x\beta(x) = x\beta(e_0)$, we get $\kappa(e) = e_0 + e$ for $e \in X - \langle e_0 \rangle$ and therefore $\kappa(e_0) = e_0$ since κ is a permutation. We conclude $\kappa(e) + \kappa(f) + \kappa(e + f) \in \langle e_0 \rangle$ for all $e, f \in X$ and even $\kappa(e) + \kappa(f) + \kappa(e + f) = e_0$ holds if $e, f, e + f \in X - \langle e_0 \rangle$. Thus $P(\mathcal{D}) \subseteq \langle e_0 \rangle \oplus Y$ by Lemma 2.3.

Assume first $n > 3$. Suppose $P(\mathcal{D}) \subset \langle e_0 \rangle \oplus Y$. Then $\dim P(\mathcal{D}) + n = \dim U(\mathcal{D})$, i.e. $\mathcal{D}$ is alternating by Corollary 2.2. Hence $\mathcal{D}_n \simeq \mathcal{H}_n$ as ambient space of $\mathcal{D}_n$ and $\mathcal{H}_n$ are the same, a contradiction. Hence $P(\mathcal{D}) = \langle e_0 \rangle \oplus Y$.

For $n = 3$ one has $\mathcal{D}_3 \simeq \mathcal{H}_3$ (cf. [7, Appendix]). Hence $\dim P(\mathcal{D}_n) = 3$. We compute $[\langle e_0 \rangle, T] = \langle \overline{e_0 \cdot e_1}, \overline{e_0 \cdot e_2} \rangle$ and assertion (a) follows. □

3.2 Quotients of Huybrechts DHOs

In this subsection we study quotients of Huybrechts DHOs. We recall their definition: Let $X = X_n = \langle e_0, \ldots, e_{n-1} \rangle$ be a n-dimensional $\mathbb{F}_2$-space, $Y = Y_n = \wedge^2(X)$ and $U = X \oplus Y$. The Huybrechts DHO of rank n has the form

$$\mathcal{H}_n = \{X(e) \mid e \in X\}$$

where

$$X(e) = \{(x, x \wedge e) \mid x \in X\},$$

in particular $X(0) = X \oplus 0$. We like to show:

Proposition 3.2. *Let $n \geq 13$. For every $k \in \{0, \ldots, n\}$ there exists a subspace $W \subseteq U(\mathcal{H}_n)$ such that $\mathcal{S} = \mathcal{H}_n / W$ is a DHO (of rank n) and*

$$\dim U(\mathcal{S}) - \dim P(\mathcal{S}) = k.$$

Lemma 3.3. *Let $\mathbf{S}_n$ be the $\mathbb{F}_2$-space of skewsymmetric $n \times n$-matrices. Let $n \geq 13$. Then $\mathbf{S}_n$ contains a subspace $\mathbf{L}_n$ such that $\dim \mathbf{L}_n \geq n$ and each nontrivial matrix in $\mathbf{L}_n$ has rank ≥ 6.*

Proof. For a fixed n denote by e_{ij} the matrix which is 1 for the position (i, j) and whose other entries are zero. For $1 \leq j < i \leq n$ set $\epsilon_{ij} = e_{ij} + e_{ji}$. Then $\{\epsilon_{ij} \mid 1 \leq j < i \leq n\}$ is a basis of $\mathbf{S}_n$. For $2 \leq j \leq n$ set

$$\mathbf{S}^j = \mathbf{S}_n^j = \langle \epsilon_{ik} \mid i > k,\ i + k = j + 1 \rangle.$$

$\mathbf{S}^j$ is the subspace in $\mathbf{S}_n$ whose matrices have their nontrivial entries in the diagonal $(j,1),(j-1,2),\dots,(1,j)$. Then

$$\dim \mathbf{S}^j = \lfloor j/2 \rfloor.$$

Let $n \geq j \geq 6$ and let $\mathbf{L}^j$ be a subspace of $\mathbf{S}^j$ of maximal dimension with respect that $w(T) \geq 3$ for $0 \neq T \in \mathbf{L}^j$. Here $w(T)$ is the Hamming weight of T with respect to the basis $\{\epsilon_{j-i+1,i} \mid 1 \leq i \leq \lfloor j/2 \rfloor\}$. Let m be the minimal number with $\lfloor j/2 \rfloor \leq 2^m - 1$. Then

$$\dim \mathbf{L}^j \geq \lfloor j/2 \rfloor - m.$$

(Let $k = \lfloor j/2 \rfloor$ and $0 \neq x_1,\dots,x_k \in \mathbb{F}_2^m$ column vectors. Set $H = (x_1,\dots,x_k)$ and consider H as the control matrix of a $\mathbb{F}_2$-code $\mathcal{C}$. Then $\mathcal{C}$ has dimension $\geq k-m$ and minimal weight ≥ 3.) Set $\mathbf{L}_n = \mathbf{L}_n^6 \oplus \cdots \oplus \mathbf{L}_n^n$. Then

$$\dim \mathbf{L}_n \geq 1+1+1+1+2+2+3+3 = 14$$

as $\dim \mathbf{L}^j \geq 1$ for $j = 6,7,8,9$, ≥ 2 for $j = 10,11$ and ≥ 3 for $j = 12,13$. As $\dim \mathbf{L}_{n+1} = \dim \mathbf{L}_n + \dim \mathbf{L}_{n+1}^{n+1}$ one has $\dim \mathbf{L}_n \geq n$ for all $n \geq 13$. Let $0 \neq T \in \mathbf{L}_n$. Write $T = T_6 + \cdots + T_n$ with $T_j \in \mathbf{L}^j$. Let j be maximal with $T_j \neq 0$. Then all entries of T below the diagonal $(j,1),(j-1,2),\dots,(1,j)$ are trivial and in this diagonal at least 6 entries are 1. Hence T has rank ≥ 6. □

Proof. (of Proposition 3.2) One knows that one can identify Y_n with $\mathbf{S}_n$ such that an element $0 \neq v \wedge w \in Y_n$ is identified with an element in $\mathbf{S}_n$ of rank 2. Consider the element $(x,y) \in X_n \oplus Y_n = U(\mathcal{H}_n)$. If $(x,y) = (x, x \wedge e) \in X(e)$ then $\operatorname{rk} y \leq 2$ and if $(x,y) \in X + X'$, $X, X' \in \mathcal{H}_n$ then $\operatorname{rk} y \leq 4$. Assume $\operatorname{rk} y \geq 6$. Then (x,y) lies not in $X + X'$, for any pair $X, X' \in \mathcal{H}_n$. Pick a subspace $\mathbf{L}_n$ of $\mathbf{S}_n$ as in Lemma 3.3. Let $y_0,\dots,y_{n-1} \in \mathbf{L}_n \subseteq Y_n \doteq \mathbf{S}_n$ be linear independent elements. Set $u_i = (e_i, y_i)$, $0 \leq i < n$ and $W_k = \langle u_0,\dots,u_{k-1} \rangle$. Then $\dim W_k = k$, $P(\mathcal{H}_n) \cap W_k = Y_n \cap W_k = 0$ and $(X + X') \cap W_k = 0$ for $X, X' \in \mathcal{H}_n$. So $\mathcal{S} = \mathcal{H}_n / W_k$ is a DHO by [16, Prop. 2.11] and $P(\mathcal{S}) = (P(\mathcal{H}_n) + W_k)/W_k$ has rank $\binom{n}{2}$. So $\dim U(\mathcal{S}) - \dim P(\mathcal{S}) = \binom{n+1}{2} - k - \binom{n}{2} = n - k$. Interchanging the roles of $n-k$ and of k we get the assertion. □

Remark 3.4. (a) Let n be odd and $\mathbf{M}_n$ a DHO set of a bilinear *orthogonal* DHO. Such DHOs exist for all odd n by [9]. Then $\mathbf{M}_n$ is a subspace of rank n of $\mathbf{S}_n$ whose nontrivial elements have rank $n-1$. So for $n \geq 7$, n odd and arguing as in the proof of Proposition 3.2 with $\mathbf{M}_n$ in the role of $\mathbf{L}_n$ we get the assertion of Proposition 3.2 with less effort.

(b) It is not difficult for $8 \leq n \leq 12$, n even, to find by an obvious random search n-subspace $\mathbf{M}_n$ of $\mathbf{S}_n$ such that nontrivial elements have rank ≥ 6. So (using (a)) the assertion of Proposition 3.2 holds even for $n \geq 7$.

(c) Presumably it is easy to mimic the arguments of this Subsection to obtain the analogue of Proposition 3.2 for the Buratti-Del Fra DHOs. In fact computations (see Remark 7.5 below) indicate that $\mathcal{D}_n$, $n \geq 5$ should have for every $0 \leq k \leq n-1$ quotients $\mathcal{S}$ with $\dim U(\mathcal{S}) - \dim P(\mathcal{S}) = k$.

4 Substructures

The most straightforward definition of a substructure of a DHO is:

Definition 4.1. Let $2 \le n' \le n$ and $\mathcal{S}$ a DHO of rank n over $\mathbb{F}_q$. A DHO $\mathcal{S}'$ of rank n' in $U(\mathcal{S})$ is a *subDHO* of $\mathcal{S}$ if every space $Z' \in \mathcal{S}'$ is contained in a subspace $Z \in \mathcal{S}$.

Our notation corresponds to the notions of [17]. In [17, Def. 1.1] Yoshiara defines subDAs (sub-dual arcs). A subDHO is a subDA which is even a DHO.

Keep the notation of the definition. By the definition of a DHO the space Z which contains Z' is uniquely determined. We define (in accordance with the conventions of [17])

$\mathcal{S}(\mathcal{S}')$ as the subset of elements of $\mathcal{S}$ which contain members of $\mathcal{S}'$.

For a DHO $\mathcal{S}$ and a subspace A of $U(\mathcal{S})$ define

$$\mathcal{S}(A) = \{Z \in \mathcal{S} \mid \dim Z \cap A > 0\}$$

and

$$\mathcal{S}[A] = \{Z' \mid Z \in \mathcal{S}(A)\}$$

with

$$Z' = \langle S \cap Z \mid Z \neq S \in \mathcal{S}(A)\rangle.$$

Lemma 4.2. *Let $\mathcal{S}$ be a DHO of rank n over $\mathbb{F}_q$. Let $X' \subseteq X \in \mathcal{S}$ be a subspace of rank $n' \ge 2$. Then there exists at most one subDHO of rank n' that contains X', namely $\mathcal{S}[X']$.*

Proof. Let $\mathcal{S}'$ be a subDHO containing X'. Then $\mathcal{S}(\mathcal{S}') = \{X\} \cup \{Z \in \mathcal{S}(\mathcal{S}') - \{X\} \mid Z \cap X' \neq 0\} = \mathcal{S}(X')$. So for $Z' \in \mathcal{S}'$ we get $Z' = \{Z' \cap Z_1' \mid Z_1' \in \mathcal{S}' - \{Z'\}\} = \{Z \cap Z_1 \mid Z_1 \in \mathcal{S}(\mathcal{S}') - \{Z\}\} \in \mathcal{S}[X']$ showing the claim. □

If $q = 2$ then $Z' \in \mathcal{S}[X']$ can be written also as

$$Z' = \langle S \curlywedge Z \mid S \in \mathcal{S}(X')\rangle$$

and if $\mathcal{S}[X']$ is a subDHO we can even write

$$Z' = \{S \curlywedge Z \mid S \in \mathcal{S}(X')\}.$$

The behavior of subDHOs under homomorphisms is described by:

Lemma 4.3. *Let $\phi : U(\mathcal{S}) \to U(\mathcal{S}')$ be a covering map from the DHO $\mathcal{S}$ to the quotient $\mathcal{S}'$.*

(a) Let $\mathcal{T}$ be a subDHO of rank m of $\mathcal{S}$. Then $\mathcal{T}\phi = \{S\phi \mid S \in \mathcal{T}\}$ is a subDHO of $\mathcal{S}'$ of rank m.

(b) Let $\mathcal{T}'$ be a subDHO of rank m of $\mathcal{S}'$. Then $\mathcal{T}'\phi^{-1} = \{S\phi^{-1} \mid S \in \mathcal{T}'\}$ is a subDHO of $\mathcal{S}$ of rank m.

Here $A\phi^{-1}$ denotes the pre-image of the object A from the codomain of ϕ. The Lemma follows immediately from the fact that the covering map is injective on the members of $\mathcal{S}$.

Example 4.4. Consider the Huybrechts DHO $\mathcal{H}_n = \{X(e) \mid e \in X\}$ of rank n, where $X(e) = \{(x, x \wedge e) \mid x \in X\}$. Let $X' = \langle e_0, \ldots, e_{k-1}\rangle$ and $X'(0) = \{(x, 0) \mid x \in X'\}$. Then

$$\mathcal{H}_n(X'(0)) = \{X(e) \mid e \in X'\}.$$

Set $X'(e) = \{X(e) \curlywedge Z \mid Z \in \mathcal{H}_n(X'(0))\}$. Then $X'(e) = \{(x, x \wedge e) \mid x \in X'\}$ and $\mathcal{S}' = \{X'(e) \mid X(e) \in \mathcal{H}_n(X'(0))\}$ is isomorphic to $\mathcal{H}_k$. As $\mathrm{Aut}(\mathcal{H}_n)$ acts doubly transitive on $\mathcal{H}_n$ and the stabilizer of an $X \in \mathcal{H}_n$ acts transitively on the set of k-spaces we can state:

(*) *Let $X \in \mathcal{H}_n$ and X' a subspace of X of dimension $\dim X' = k \geq 2$. Then $\mathcal{H}_n[X']$ is subDHO of $\mathcal{H}_n$ isomorphic to $\mathcal{H}_k$.*

Let $\mathcal{S}$ be a quotient of $\mathcal{H}_n$. By Lemma 4.3 for every subspace $X' \subseteq X \in \mathcal{S}$ the set $\mathcal{S}[X']$ is a subDHO of rank $\dim X'$.

We take from Yoshiara [17, Def. 1.1] the following definition.

Definition 4.5. Let $\mathcal{S}$ be a DHO over $\mathbb{F}_2$ with subDHOs $\mathcal{T}_1, \ldots, \mathcal{T}_M$ of rank m. We say that $\mathcal{S}$ *is the disjoint union of* $\mathcal{T}_1, \ldots, \mathcal{T}_M$ if

$$\mathcal{S} = \mathcal{S}(\mathcal{T}_1) \cup \cdots \cup \mathcal{S}(\mathcal{T}_M)$$

is a partition of $\mathcal{S}$. We write in this case

$$\mathcal{S} = \mathcal{T}_1 \sqcup \cdots \sqcup \mathcal{T}_M.$$

It is convenient for the next result to define a *subDHO of rank* 1 as a 1-space of the form $X \curlywedge Z$, $X, Z \in \mathcal{S}$, $X \neq Z$.

Theorem 4.6. *Let $\mathcal{S}$ be a DHO over $\mathbb{F}_2$ of rank n. Let Q be a subspace of $U(\mathcal{S})$ which contains $P(\mathcal{S})$ such that*

$$m = n + \dim Q - \dim U(\mathcal{S}) > 0.$$

The following hold:

(a) There exist $M = 2^{n-m}$ subDHOs $\mathcal{T}_1, \ldots, \mathcal{T}_M$ of rank m such that

$$\mathcal{S} = \mathcal{T}_1 \sqcup \cdots \sqcup \mathcal{T}_M.$$

(b) For all $1 \leq j \leq M$ we have

$$\mathcal{T}_j = \{X \cap Q \mid X \in \mathcal{S}(\mathcal{T}_j)\}.$$

(c) *Suppose $X' \in \mathcal{T}_1$ is contained in X. Then the coset decomposition of X modulo X' can be written in the form*

$$X = \bigcup_{i=1}^{M} X' + v_i,$$

such that for all $1 \leq j \leq M$ we have

$$\{X \curlywedge Z \mid Z \in \mathcal{S}(\mathcal{T}_j)\} = X' + v_j.$$

Proof. Let X be in $\mathcal{S}$. By Lemma 2.1 $U(\mathcal{S}) = Q + X$. Hence

$$m = \dim X \cap Q = \dim X + \dim Q - \dim U(\mathcal{S}) = n + \dim Q - \dim U(\mathcal{S}).$$

We define the symmetric relation $\sim$ on $\mathcal{S}$ and write $X \sim Z$ if $X = Z$ or if $X \neq Z$ and $X \curlywedge Z \subseteq Q$. Note, that $\mathcal{S}(X \cap Q) = \{Z \in \mathcal{S} \mid X \sim Z\}$.

CLAIM: $\sim$ is an equivalence relation on $\mathcal{S}$.

If $m = 1$ then for any X there exists precisely one X' with $X \cap Q = \langle X \cap X' \rangle$, i.e. the claim holds.

So assume $m > 1$. It is enough to show that $X_1 \sim X_2$ for $X_1, X_2 \in \mathcal{S}(X \cap Q)$. We know $X \curlywedge X_1, X \curlywedge X_2 \in Q$. Therefore

$$X_1 \curlywedge X_2 = u(X, X_1, X_2) + X \curlywedge X_1 + X \curlywedge X_2 \in Q$$

as $P(\mathcal{S}) \subseteq Q$. Thus $X_1 \sim X_2$ and the claim follows.

Moreover, every equivalence class has the form $\mathcal{S}(X \cap Q)$ where X is any member from this class. We have seen $X \cap Q = \{X \curlywedge Z \mid Z \in \mathcal{S}(X \cap Q)\}$. So $\mathcal{S}[X \cap Q]$ is a collection of m-spaces. Since $\mathcal{S}$ is a DHO and by the definition of $\sim$ we conclude that $\mathcal{T} = \mathcal{S}[X \cap Q]$ is a subDHO of rank m. Choose $X_1, \ldots, X_M$ such that

$$\mathcal{S} = \mathcal{S}(X_1 \cap Q) \cup \cdots \cup \mathcal{S}(X_M \cap Q)$$

is the partition into the equivalence classes and set $\mathcal{T}_i = \mathcal{S}[X_i \cap Q]$. Then $2^m = |\mathcal{T}_i| = |\mathcal{S}(X_i \cap Q)|$ showing $M = 2^{n-m}$. Assertions (a) and (b) follow.

Let $X' = X \cap Q \in \mathcal{T}_1$, $X \in \mathcal{S}(\mathcal{T}_1)$ and pick $Z_1, Z_2 \in \mathcal{S}(\mathcal{T}_j)$. Then

$$X \curlywedge Z_1 + X \curlywedge Z_2 = u(X, Z_1, Z_2) + Z_1 \curlywedge Z_2 \in X \cap Q = X'.$$

So $X \curlywedge Z_1, X \curlywedge Z_2$ lie in the same coset modulo X'. Hence

$$\{X \curlywedge Z \mid Z \in \mathcal{S}(\mathcal{T}_j)\} = X' + v_j,$$

for some $v_j \in X$. Assertion (c) holds too. □

Example 4.7. Let $n \geq 3$, V be a n-dimensional $\mathbb{F}_2$-space and $U = S^2(V)$ the symmetric square of V. Define in U the n-spaces $X(0) = \{x^2 \mid x \in V\}$ and $X(e) = \{x \cdot e \mid x \in V\}$ for $0 \neq e \in V$. Then $\mathcal{V} = \mathcal{V}_n = \{X(e) \mid e \in V\}$ is a DHO, the *Veronesean DHO* of rank n [16, Sec. 5.2].

Observe that $X(0) \curlywedge X(e) = e^2$ and $X(e) \curlywedge X(f) = e \cdot f$ for $0 \neq e, f \in V$. Let $V' \subset V$ be a $(n-1)$-space, $X' = \{x^2 \mid x \in V'\}$, then this observation shows that the set $\mathcal{V}' = \mathcal{V}[X']$ is a subDHO isomorphic to $\mathcal{V}_{n-1}$. However $\mathcal{V}$ is not the disjoint union of $\mathcal{V}'$ with an other subDHO:

Otherwise if $\mathcal{T}$ is such a subDHO then $\mathcal{V}(Z') = \{X(e) \mid e \in V - V'\}$ for $Z' \in \mathcal{T}$. Say $Z' \subseteq X(e_0)$, $e_0 \in V - V'$. Then $Z' = X(e_0) - \{X(e_0) \curlywedge X(e) \mid e \in V'\} = \{X(e_0) \curlywedge X(e) \mid e \in V - V'\}$. However $\{e_0 \cdot e \mid e \in V - V', e \neq e_0\}$ generates $X(e_0)$ which is an n-space. This contradiction shows our claim.

For $n \geq 4$ Taniguchi defines over the n-space V a distortion $\mathcal{T}_n$ of the Veronesean DHO $\mathcal{V}_n$, the *Taniguchi DHO* of rank n (see for instance [15]). *Certain* $(n-1)$-spaces V' of V define subDHOs isomorphic to $\mathcal{T}_{n-1}$ which are again not members of a disjoint union of subDHOs of $\mathcal{T}_n$. We leave the somewhat more elaborate verification to the reader.

4.1 Hyperplanes and subDHOs

The next Proposition is a slight generalization of [17, Prop. 1.2(2)].

Proposition 4.8. *Let $\mathcal{S}$ be a DHO over $\mathbb{F}_2$ of rank n and $\mathcal{T}$ a subDHO of rank $n-1$. The following hold:*

(a) For every $Z \in \mathcal{S} - \mathcal{S}(\mathcal{T})$ we have $U(\mathcal{S}) = U(\mathcal{T}) + Z$.

(b) For $X' \in \mathcal{T}$ and $Z \in \mathcal{S} - \mathcal{S}(\mathcal{T})$ we have $X' \cap Z = 0$.

Proof. Set $Q = U(\mathcal{T}) + Z$. Let $X' \in \mathcal{T}$ and $X' \subseteq X \in \mathcal{S}(\mathcal{T})$. Then $X \curlywedge Z \in X - X'$, i.e. $X = X' \oplus (X \cap Z) \subseteq Q$, i.e. $U(\mathcal{S}(\mathcal{T})) \subseteq Q$. For $Y \in \mathcal{S} - \mathcal{S}(\mathcal{T})$ set $\bar{Y} = \{Y' \curlywedge Y \mid Y' \in \mathcal{S} - \mathcal{S}(\mathcal{T})\}$. As $\mathcal{S} = \mathcal{S}(\mathcal{T}) \cup (\mathcal{S} - \mathcal{S}(\mathcal{T}))$ is a partition into 2^{n-1}-sets we get $|\bar{Y}| = 2^{n-1}$. And therefore $Y_0 = \{Y \curlywedge X \mid X \in \mathcal{S}(\mathcal{T})\} = Y - \bar{Y}$ is a set of size 2^{n-1} of vectors $\neq 0$. In particular $Y = \langle Y_0 \rangle$. On the other hand $Y_0 \subset U(\mathcal{S}(\mathcal{T})) \subseteq Q$ showing $Y \subseteq Q$. We conclude $Q = U(\mathcal{S})$ and (a) follows. Assertion (b) is trivial. □

Definition 4.9. Let $\mathcal{S}$ be a DHO over $\mathbb{F}_2$ of rank n. Assume H is a hyperplane of $U(\mathcal{S})$ and $\mathcal{T}$ a subDHO of rank $n-1$. We say that $\mathcal{T}$ *is induced by* H or *induced by a hyperplane* if $\mathcal{T} = \{X \cap H \mid X \in \mathcal{S}(\mathcal{T})\}$ and say that H *induces a subDHO* if $\mathcal{S}$ contains a 2^{n-1}-set $\mathcal{S}'$ such that $\{X \cap H \mid X \in \mathcal{S}'\}$ is a subDHO of rank $n-1$.

Let X, Y be finite dimensional $\mathbb{F}_2$-spaces, $\beta : X \to \mathrm{Hom}(X, Y)$ be a monomorphism which defines a bilinear DHO $\mathcal{S}_\beta$ in $U = X \oplus Y$. Define $\beta^o : X \to \mathrm{Hom}(X, Y)$ by $e\beta^o(x) = x\beta(e)$, $x, e \in X$. Then β^o defines the *opposite DHO* $\mathcal{S}_{\beta^o}$ in $U = X \oplus Y$ (see [3, Sec. 3]).

Example 4.10. Let X, Y be finite dimensional $\mathbb{F}_2$-spaces, $\dim X = n$ and let $\beta : X \to \mathrm{Hom}(X, Y)$ be a monomorphism which defines a bilinear DHO $\mathcal{S}_\beta$. Set $\overline{X} = \mathbb{F}_2 \oplus X$ and $\overline{Y} = X \oplus Y$. For $e \in X$ define two subspaces of $\overline{X} \oplus \overline{Y}$ by

$$X(0, e) = \{(b, be, be + x, (be + x)\beta(e)) \mid (b, x) \in \overline{X}\},$$

$$X(1,e) = \{(b, be+x, be, (be+x)\beta^o(e)) \mid (b,x) \in \overline{X}\},$$

and set $\overline{\mathcal{S}}_\beta = \{X(a,e) \mid (a,e) \in \overline{X}\}$. The set $\overline{\mathcal{S}}_\beta$ is a DHO of rank $n+1$ in $\overline{X} \oplus \overline{Y}$, the *extension of the bilinear DHO* $\mathcal{S}_\beta$ (see [7] and [12]).

Obviously the hyperplane $H = 0 \oplus X \oplus \overline{Y}$ induces the subDHOs $\mathcal{S}_\beta \simeq \mathcal{T}_\beta \subseteq \overline{Y}$ and $\mathcal{S}_{\beta^o} \simeq \mathcal{T}_{\beta^o} \subseteq 0 \oplus X \oplus 0 \oplus Y$ and $\overline{\mathcal{S}}_\beta = \mathcal{T}_\beta \sqcup \mathcal{T}_{\beta^o}$.

Lemma 4.11. *Let $\mathcal{S}$ be a DHO of rank n over $\mathbb{F}_2$ and let $\mathcal{T}$ be a subDHO of rank $n-1$. Suppose that there exists $Z \in \mathcal{S} - \mathcal{S}(\mathcal{T})$ such that:*

(i) There is a hyperplane Z_0 of Z such that $X \curlywedge Z \in Z - Z_0$ for all $X \in \mathcal{S}(\mathcal{T})$.

(ii) $U(\mathcal{T}) + Z_0$ is a proper subspace of $U(\mathcal{S})$.

Then $\mathcal{T}$ is induced by a hyperplane.

Proof. By Proposition 4.8 one has $U(\mathcal{T}) + Z_0 \subset U(\mathcal{S}) = U(\mathcal{T}) + Z$. So $H = U(\mathcal{T}) + Z_0$ is a hyperplane of $U(\mathcal{S})$ and $Z - Z_0 = \{Z \curlywedge X \mid X \in \mathcal{S}(\mathcal{T})\} \subset U(\mathcal{S}) - H$. This implies $\mathcal{T} = \{H \cap X \mid X \in \mathcal{S}(\mathcal{T})\}$. □

Proposition 4.12. *Let $\mathcal{S}$ be a DHO of rank n over $\mathbb{F}_2$ and let $\mathcal{T}$ be a subDHO of rank $n-1$ that is induced by the hyperplane H of $U(\mathcal{S})$. Set $\mathcal{S}' = \mathcal{S} - \mathcal{S}(\mathcal{T})$. The following hold:*

(a) Let X be in $\mathcal{S}(\mathcal{T})$ and Z in $\mathcal{S}'$. Then $X \curlywedge Z \in U(\mathcal{S}) - H$.

(b) $\mathcal{T}' = \{Z \cap H \mid Z \in \mathcal{S}'\}$ is a subDHO of rank $n-1$.

(c) $P(\mathcal{S}) \subseteq H$.

Proof. Let $X' \in \mathcal{T}$ be contained in $X \in \mathcal{S}(\mathcal{T})$ and $Z \in \mathcal{S}'$. Then $X' = X \cap H$ and $X = X' \cup (X' + X \curlywedge Z)$ is the coset decomposition of X modulo X'. In particular $X \curlywedge Z \in X - (X \cap H)$, i.e. (a) holds.

The set $\bar{Z} = \{Z \curlywedge X \mid X \in \mathcal{S}(\mathcal{T})\}$ is a set of nontrivial vectors of size 2^{n-1} which lies in $U(\mathcal{S}) - H$, in particular $Z = \langle \bar{Z} \rangle$. Hence $\{Z \curlywedge Z' \mid Z' \in \mathcal{S}'\} = Z - \bar{Z} \subset H$. We deduce

$$Z \cap H = \{Z \curlywedge Z' \mid Z' \in \mathcal{S}'\}.$$

Assertion (b) follows.

By (b) $U(\mathcal{T}) + U(\mathcal{T}') \subseteq H$. This shows $u(X,Y,Z) \in H$ if $\{X,Y,Z\}$ lies in $\mathcal{S}(\mathcal{T})$ or $\mathcal{S}(\mathcal{T}')$. Suppose now $X, Y \in \mathcal{S}(\mathcal{T})$, $Z \in \mathcal{S}(\mathcal{T}')$. Then $X \curlywedge Y \in H$ and $X \curlywedge Z, Y \curlywedge Z \in U(\mathcal{S}) - H$. Since H is a hyperplane: $X \curlywedge Z + Y \curlywedge Z \in H$. So again $u(X,Y,Z) \in H$. Arguing by symmetry we conclude that assertion (c) holds. □

Corollary 4.13. *Let $\mathcal{S}$ be a DHO over $\mathbb{F}_2$ and let $\mathcal{H}$ be the set of hyperplanes which induce a subDHO. Then*

$$P(\mathcal{S}) = \bigcap_{H \in \mathcal{H}} H.$$

Proof. By assertion (c) of Proposition 4.12 $P(\mathcal{S}) \subseteq H$ for every $H \in \mathcal{H}$. So $P(\mathcal{S}) \subseteq \bigcap_{H\in\mathcal{H}} H$.

Let $\mathcal{H}'$ be the set of hyperplanes containing $P(\mathcal{S})$. Then $P(\mathcal{S}) = \bigcap_{K\in\mathcal{H}'} K$. By assertion (b) of Theorem 4.6 every $K \in \mathcal{H}'$ (take K in the role of Q) lies in $\mathcal{H}$. Hence $\bigcap_{H\in\mathcal{H}} H \subseteq P(\mathcal{S})$ and the claim follows. □

Corollary 4.14. *Let $\mathcal{S}$ be a DHO over $\mathbb{F}_2$ of rank n and $\mathcal{T}$ be a subDHO of rank $n-1$. Equivalent are:*

(a) $\mathcal{T}$ is induced by a hyperplane.

(b) $H = P(\mathcal{S}) + U(\mathcal{T})$ is a proper subspace (and then even a hyperplane).

Proof. Let $\mathcal{T}$ be induced by H. Then $P(\mathcal{S}) \subseteq H$ by Proposition 4.12. By Lemma 2.1 $U(\mathcal{S}) = P(\mathcal{S}) + X$ for $X \in \mathcal{S}(\mathcal{T})$ and by Theorem 4.6 $\mathcal{T} = \mathcal{S}[X']$, $X' = X \cap H$. So $P(\mathcal{S}) + X' \subseteq P(\mathcal{S}) + U(\mathcal{T}) \subseteq H \subset P(\mathcal{S}) + X = U(\mathcal{S})$. Hence $H = P(\mathcal{S}) + U(\mathcal{T})$.

Now assume that H is a proper subspace of $U(\mathcal{S})$. Suppose $X' \in \mathcal{T}$, $X' \subset X \in \mathcal{S}$. By Lemma 2.1 then $P(\mathcal{S}) + X' \subseteq H$ has codimension ≤ 1 in $U(\mathcal{S}) = P(\mathcal{S}) + X$. So H is a hyperplane which induces $\mathcal{T}$. □

Remark 4.15. Suppose the binary DHO $\mathcal{S} = \mathcal{T}_1 \sqcup \mathcal{T}_2$ of rank n is the disjoint union of two subDHOs of rank $n-1$.

(a) With $Z \in \mathcal{S}(\mathcal{T}_2)$ and $\mathcal{T} = \mathcal{T}_1$ assumption (i) of Lemma 4.11 is fulfilled. So assumption (ii) holds too, iff $P(\mathcal{S})$ is a proper subspace of $U(\mathcal{S})$. The assumptions of Yoshiara [17, Theorem 1.3] imply $\dim U(\mathcal{T}_1) + n = \dim U(\mathcal{S})$, which implies (ii) too. Thus the assumptions of Yoshiaras Theorem force that the radical is a proper subspace of the ambient space.

(b) Let $W \subset U(\mathcal{S})$ be a subspace such that $\mathcal{S}/W$ is a quotient of $\mathcal{S}$ and H be a hyperplane. If $W \not\subseteq H$, then $U(\mathcal{S}/W) = U(\mathcal{T}/W) + (Z_0 + W)/W$ for $Z \in \mathcal{S}$ and $Z_0 = Z \cap H$ (see Lemma 4.11) but still $\mathcal{S}/W = \mathcal{T}_1/W \sqcup \mathcal{T}_2/W$ holds although $\mathcal{T}_i/W$ is not induced by a hyperplane. By Proposition 3.2 there are many quotients of Huybrechts DHOs with this property.

DHOs which are disjoint unions of proper subDHOs and which have not a proper radical (as in (b) of the preceding Remark) are indeed always proper quotients:

Corollary 4.16. *Let $\mathcal{S}$ be a simply connected DHO over $\mathbb{F}_2$ of rank n which is the disjoint union of subDHOs $\mathcal{T}_1$ and $\mathcal{T}_2$ of rank $n-1$. Then $\mathcal{T}_1$ and $\mathcal{T}_2$ are hyperplane-induced.*

Proof. Let $U = U(\mathcal{S}) \oplus \mathbb{F}_2$. For $X \in \mathcal{S}$ we define a set $\tilde{X} \subseteq U$ by

$$\tilde{X} = \bar{X} + \{x + 1 \mid x \in X - \bar{X}\}$$

when $X \in \mathcal{S}$ lies in $\mathcal{S}(\mathcal{T}_i)$ and $\bar{X} \in \mathcal{T}_i$. One checks that $\tilde{X}$ is a $\mathbb{F}_2$-space with the same dimension as X. Furthermore, since $X \curlywedge X' \in X - \bar{X}$ for $X \in \mathcal{S}(\mathcal{T}_1)$

and $X' \in \mathcal{S}(\mathcal{T}_2)$ we see that $\tilde{\mathcal{S}} = \{\tilde{X} \mid X \in \mathcal{S}\}$ is a DHO in U. Define an epimorphism $\phi : U \to U(\mathcal{S})$ by $(x + \alpha)\phi = x$ for $x \in X$, $\alpha \in \mathbb{F}_2$. We observe $(\tilde{X} \curlywedge \tilde{X}')\phi = X \curlywedge X'$. So ϕ is a covering map.

Suppose that $\mathcal{T}_1$ is not hyperplane-induced. Assumption (i) of Lemma 4.11 holds as $\mathcal{S} = \mathcal{T}_1 \sqcup \mathcal{T}_2$. Hence (ii) of Lemma 4.11 is violated, i.e. $U(\mathcal{S}) = U(\mathcal{T}_1) + \bar{Z}$ for some $Z \in \mathcal{S}(\mathcal{T}_2)$. Therefore $U(\mathcal{S}) \subseteq U(\tilde{\mathcal{S}})$. But as $\tilde{X} \not\subseteq U(\mathcal{S})$ we get $U = U(\tilde{\mathcal{S}})$. So $\tilde{\mathcal{S}}$ is a proper cover of $\mathcal{S}$, a contradiction. □

Remark 4.17. The Veronesean DHO $\mathcal{V}_n$ and $\mathcal{T}_n$ are both simply connected and their radical coincides with the ambient space. Corollary 4.16 gives an immediate (non-computational) explanation for the claims in Example 4.7.

5 The radical of bilinear DHOs

The next Theorem improves Lemma 2.3. Afterwards we discuss consequences of this Theorem.

Theorem 5.1. *Let X, Y be $\mathbb{F}_2$-spaces, $\beta : X \to \mathrm{Hom}(X, Y)$ be a monomorphism which defines a bilinear DHO $\mathcal{S} = \mathcal{S}_\beta$ of rank $n \geq 4$ with ambient space $U = X \oplus Y$. Set $X_\kappa = \langle \kappa(e) + \kappa(f) + \kappa(e+f) \mid 0 \neq e, f \in X \rangle$. Then $P(\mathcal{S}) = X_\kappa \oplus Y$.*

Proof. We prove the Theorem by induction on the rank. Basis for induction is $n = 4$. The bilinear DHOs of rank 4 are classified in [1]. The inspection of these examples shows that the Theorem holds for $n = 4$.

$n \Rightarrow n+1$: Let $\mathcal{S} = \mathcal{S}_\beta$ be a bilinear DHO of rank $n + 1 > 4$. Set $P = P(\mathcal{S})$, $U = U(\mathcal{S})$ and let T be the standard translation group. By the proof of Lemma 2.3 it suffices to show that $Y \subseteq P$. If $U = P + Y$ then (as $C_U(T) = Y$)

$$Y = [U, T] = [P + Y, T] = [P, T] \subseteq P$$

and we are done.

So assume $P + Y \neq U$. Hence there exists a hyperplane H of U which contains $P + Y$. By Proposition 4.12 H induces two subDHOs $\mathcal{T}_0$ and $\mathcal{T}_1$ (of rank n) such that $\mathcal{S} = \mathcal{T}_0 \sqcup \mathcal{T}_1$. Since $Y \subseteq H$ this hyperplane is invariant under T. In particular T contains a subgroup T_0 of index 2 which fixes $\mathcal{S}(\mathcal{T}_0)$ and $\mathcal{S}(\mathcal{T}_1)$ and which acts regularly on both sets. This implies that $\mathcal{T}_0$ and $\mathcal{T}_1$ are bilinear and T_0 induces on both DHOs a translation group. Let $\tau_0 \in T - T_0$. Then τ_0 interchanges $\mathcal{S}(\mathcal{T}_0)$ and $\mathcal{S}(\mathcal{T}_1)$ and fixes $X \curlywedge X'$, $X \in \mathcal{S}(\mathcal{T}_0)$ and $X' = X\tau_0 \in \mathcal{S}(\mathcal{T}_1)$. By Proposition 4.8 $H = (X' \cap H) + U(\mathcal{T}_0)$ and $U = X \curlywedge X' \oplus H$. As $C_U(\tau_0) \cap X = X \curlywedge X'$ we have $|Y_D| = |X \cap H|$ for $Y_D = [X, \tau_0] \subseteq Y$ and

$$(X \cap H) \oplus (X' \cap H) = (X \cap H) \oplus Y_D = (X' \cap H) \oplus Y_D.$$

Set $Y_P = Y \cap P$. By induction $Y_i = [U(\mathcal{T}_i), T_0] \subseteq Y_P$ for $i = 0, 1$. So

$$H = (X' \cap H) + U(\mathcal{T}_0) = (X' \cap H) + (X \cap H) + Y_0 = (X \cap H) + Y_D + Y_P.$$

In particular $Y = Y_D + Y_P$ as $H = X_H \oplus Y$ where $X_H = X \cap H$.

We now introduce coordinates for U. Set $\langle e_0 \rangle = X \curlywedge X'$. We choose a basis $\{e_1, \dots e_n\}$ of X_H. Then there exist $y_i \in Y$ (even $y_i \in Y_D$) such that $e_i' = e_i \tau_0 = e_i + y_i$ and that $Y = Y_C \oplus Y_P$ where $Y_C = \langle y_1, \dots, y_d \rangle$ and $\{y_{d+1}, \dots, y_n\} \subset Y_P$ where $Y_D \cap Y_P = \langle y_{d+1}, \dots, y_n \rangle$. We claim $d = 0$. Then $Y = Y_P$ and we are done.

So assume $d \geq 1$. There exist $y_{n+1}, \dots, y_m \in Y_P$ such that

$$\{e_0, e_1, \dots, e_n, y_1, \dots, y_d, y_{d+1}, \dots, y_m\}$$

is a basis of U. In particular

$$U = \langle e_0 \rangle \oplus X_C \oplus X_P \oplus Y_C \oplus Y_P$$

with $X_C = \langle e_1, \dots, e_d \rangle$, $X_P = \langle e_{d+1}, \dots, e_n \rangle$, $Y_C = \langle y_1, \dots, y_d \rangle$. We describe therefore the elements u in U by tuples $u = (b, x_C, x_P, y_C, y_P)$ where

$$u = be_0 + \sum_{i=1}^{d} x_i e_i + \sum_{i=d+1}^{n} x_i e_i + \sum_{i=1}^{d} z_i y_i + y_P$$

where $x_C = (x_1, \dots, x_d) \in \mathbb{F}_2^d$, $x_P = (x_{d+1}, \dots, x_n) \in \mathbb{F}_2^{n-d}$, $y_C = (z_1, \dots, z_d) \in \mathbb{F}_2^d$ and where we identify $y_P \in Y_P$ with $(z_{d+1}, \dots, z_m) \in \mathbb{F}_2^k$ if $y_P = \sum_{d+1}^{m} z_i y_i$ and $m = d + k$. The elements in X have the form $(b, x_C, x_P, 0, 0)$ with $b \in \mathbb{F}_2$, $x_C \in \mathbb{F}_2^d$, $x_P \in \mathbb{F}_2^{n-d}$ and the elements in X' the form (b, x_C, x_P, x_C, x_P).

We express $\beta(e)$, $e \in X_H$ with respect to the given basis:

$$\beta(e) = \begin{pmatrix} a(e) & c(e) \\ 0 & A(e) \\ 0 & B(e) \end{pmatrix} \quad \text{where} \quad \beta_{X_H}(e) = \begin{pmatrix} 0 & A(e) \\ 0 & B(e) \end{pmatrix}$$

with $a(e) \in \mathbb{F}_2^d$, $c(e) \in \mathbb{F}_2^k$, $A(e) \in \mathbb{F}_2^{d \times k}$ and $B(e) \in \mathbb{F}_2^{(n-d) \times k}$. A typical element in $X(e) = X\tau_e$, $e \in X_H$ has the form

$$(b, x_C, x_P, ba(e), *) = (b, x_C, x_P, ba(e), bc(e) + x_C A(e) + x_P B(e)).$$

We compute for $0 \neq e \in X_H$ the elements $X \curlywedge X(e)$ and $X' \curlywedge X(e)$. We get $X \curlywedge X(e) = \langle (0, 0, \kappa(e), 0, 0) \rangle$ and

$$X' \curlywedge X(e) = \langle (1, a(e), *, a(e), *) \rangle.$$

Since $X' - H = \{X' \curlywedge Z \mid Z \in \mathcal{S}(\mathcal{T}_0)\} = \{X' \curlywedge X(e) \mid e \in X_H\}$ we see that $a : X_H \to Y_C$ is an epimorphism. Moreover

$$u = u(X, X', X(e)) = (0, a(e), *, a(e), *) \in P.$$

Since P is invariant under τ_0 we have

$$(0, 0, 0, a(e), *) = [u, \tau_0] \in Y_P.$$

As a is an epimorphism we conclude $Y \subseteq P$, a contradiction. So $d = 0$ and the proof is complete. □

Remark 5.2. Lemma 3.1 (a) shows that the assumption $n \geq 4$ in Theorem 5.1 is necessary.

Proposition 5.3. *Let X, Y be $\mathbb{F}_2$-spaces, $\beta : X \to \mathrm{Hom}(X, Y)$ be a monomorphism which defines a bilinear DHO $\mathcal{S} = \mathcal{S}_\beta$ with ambient space $U = X \oplus Y$. Denote by κ^o the kernel function of the opposite DHO. Then $\dim X_\kappa = \dim X_{\kappa^o}$ and*

$$\dim P(\mathcal{S}_\beta) = \dim P(\mathcal{S}_{\beta^o}).$$

Proof. We have $\kappa(x) + \kappa(x') \equiv \kappa(x + x') \pmod{X_\kappa}$ for all $x, x' \in X$ and by induction

$$\sum_{i=1}^{\ell} \kappa(x_i) \equiv \kappa(\sum_{i=1}^{\ell} x_i) \pmod{X_\kappa}$$

for $x_i \in X$. Set

$$D = \langle (x, \kappa(x)) \mid x \in X \rangle \subseteq X \oplus X$$

and assume $\dim X = n$.

Claim: $\dim D = n + \dim X_\kappa$.

Let $\pi : D \to X$ the projection on $X \oplus 0$. As $(0, \kappa(e) + \kappa(e) + \kappa(e + f)) = (e + f + (e + f), \kappa(e) + \kappa(e) + \kappa(e + f))$ we see $0 \oplus X_\kappa \subseteq D \cap \ker \pi$. Let $(\sum_{i=1}^{\ell} x_i, \sum_{i=1}^{\ell} \kappa(x_i))$ be an element in $\ker \pi$. Then $\sum_i x_i = 0$, so that

$$(\sum_{i=1}^{\ell} x_i, \sum_{i=1}^{\ell} \kappa(x_i)) \equiv (0, \kappa(\sum_{i=1}^{\ell} x_i)) \equiv 0 \pmod{0 \oplus X_\kappa},$$

i.e. $\sum_{i=1}^{\ell} \kappa(x_i) \in X_\kappa$ and the claim follows.

By definition of the opposite DHO we have $\kappa^{-1} = \kappa^o$. So $D = \langle (x, \kappa(x)) \mid x \in X \rangle = \langle (\kappa^o(x), \kappa^o(\kappa(x))) \mid x \in X \rangle = \langle (\kappa^o(x), x) \mid x \in X \rangle$ and by symmetry $\dim X_{\kappa^o} + n = \dim \langle (\kappa^o(x), x) \mid x \in X \rangle$. So $\dim X_{\kappa^o} = \dim X_\kappa$. The second assertion is a consequence of Theorem 5.1. □

Let $\mathcal{S}$ be a DHO of rank n with an ambient space $U = U(\mathcal{S})$ of rank $2n$. Denote by U^* the dual space of U and for a subspace $W \subseteq U$ we denote by $W^t \subseteq U^*$ the space of functionals which vanish on W. Then $\mathcal{S}^t = \{X^t \mid X \in \mathcal{S}\}$ is a set of n-spaces in V^*. If $\mathcal{S}^t$ is a DHO too we call $\mathcal{S}$ a *doubly dual dimensional hyperoval* or DDHO for short (see [3]).

We might ask the question if for a DDHO $\mathcal{S}$ the spaces $P(\mathcal{S})$ and $P(\mathcal{S}^t)$ have the same rank. The following example shows that this question has a negative answer even for the special case of bilinear DDHOs.

Example 5.4. (DHOs of Yoshiara [16, Sec. 5.5]) Let $F = \mathbb{F}_{2^n}$ and σ be a generator of $\mathrm{Gal}(F : \mathbb{F}_2)$, and let ϕ be an o-polynomial on F. For $e \in F$ define $X(e) = \{(x, x^\sigma e + x\phi(e)) \mid x \in F\}$ then $\mathcal{S} = \mathcal{S}^n_{\sigma,\phi} = \{X(e) \mid e \in F\}$ is a DHO in $F \times F$. If ϕ is a power function $\mathrm{Aut}(\mathcal{S})$ contains a cyclic group C such that C fixes $X = X(0)$ and C acts transitively on $X - \{0\}$ (see [13, Thm. 1.1]). Then

$X \cap P(\mathcal{S})$ is a C-invariant space. So either $X \cap P(\mathcal{S}) = 0$ so that by Lemma 2.1 and Corollary 2.2 $\mathcal{S}$ is alternating (in this case $\sigma = \phi$) or $X \subseteq P(\mathcal{S})$ and hence $P(\mathcal{S}) = U(\mathcal{S})$ by Lemma 2.1.

Consider a bilinear DHO Yoshiara DHO $\mathcal{S} = \mathcal{S}^n_{\sigma,\phi}$; in this case ϕ is a generator of $\mathrm{Gal}(F : \mathbb{F}_2)$ too and $\mathcal{S} = \mathcal{S}_\beta$ with $x\beta(e) = xe^\phi + x^\sigma e$. Define a symplectic form $A : F \times F \to \mathbb{F}_2$ by $A(x, y) = \mathrm{T}(xy)$ (here $\mathrm{T} : F \to \mathbb{F}_2$ is the absolute trace on F). Then $\mathcal{S}^t$ is isomorphic to $\mathcal{S}_{\beta^*}$ where the operator $\beta^*(e)$ is adjoint to $\beta(e)$ with respect to A (so $A(x, y\beta(e)) = A(x\beta^*(e), y)$), see [3, Sec. 3]). A computation shows that $\mathcal{S}$ is a DDHO only if n is odd and that $\mathcal{S}_{\beta^*} \simeq \mathcal{S}^n_{\sigma\circ\phi,\phi^{-1}}$.

So if $\mathcal{S} = \mathcal{S}^n_{\sigma,\sigma}$ is alternating then $\mathcal{S}^t \simeq \mathcal{S}^n_{\sigma^2,\sigma^{-1}}$ is not alternating if $\sigma^3 \neq 1$ and

$$\dim P(\mathcal{S}) = n \quad \text{whereas} \quad \dim P(\mathcal{S}^t) = 2n.$$

Theorem 5.1 allows a concrete description of the radical of extensions of bilinear DHOs (see Example 4.10).

Proposition 5.5. *Let X, Y be $\mathbb{F}_2$-spaces, $\beta : X \to \mathrm{Hom}(X, Y)$ be a monomorphism which defines a bilinear DHO $\mathcal{S} = \mathcal{S}_\beta$ with ambient space $U = X \oplus Y$. Let $\bar{\mathcal{S}} \subseteq \mathbb{F}_2 \oplus X \oplus X \oplus Y$ be the extension of $\mathcal{S}$.*

(a) Set $D = \langle (0, e, \kappa(e), 0) \mid e \in X \rangle$. Then

$$P(\bar{\mathcal{S}}) = D \oplus (0 \oplus 0 \oplus 0 \oplus Y).$$

(b) $\dim U(\bar{\mathcal{S}}) - \dim P(\bar{\mathcal{S}}) = \dim U(\mathcal{S}) - \dim P(\mathcal{S}) + 1$.

Proof. We use the description of the extension $\bar{\mathcal{S}}$ of $\mathcal{S}$ from Example 4.10. The automorphism group of $\bar{\mathcal{S}}$ contains an elementary abelian group $N = \langle n_{1,e}, n_{0,e} \mid e \in X \rangle$, the *extension group*, which is generated by the operators

$$n_{1,e} = \begin{pmatrix} 1 & e & & \\ & \mathbf{1} & & \\ & & \mathbf{1} & \beta(e) \\ & & & \mathbf{1} \end{pmatrix} \quad \text{and} \quad n_{0,e} = \begin{pmatrix} 1 & & e & \\ & \mathbf{1} & & \beta^o(e) \\ & & \mathbf{1} & \\ & & & \mathbf{1} \end{pmatrix}.$$

Set $P = \{(0, 0, x, y) \mid (x, y) \in P(\mathcal{S}_\beta)\}$ and $P^o = \{(0, x, 0, y) \mid (x, y) \in P(\mathcal{S}_{\beta^o})\}$. Any $u(X_1, X_2, X_3) \in P(\bar{\mathcal{S}})$ is conjugate under N to an element of the form $u(X(b, 0), X(b, e), X(b, f))$ or $u(X(0, 0), X(1, 0), X(b, e))$, $b = 0, 1$.

Elements of the first kind lie in P or P^o and by Theorem 5.1 and Proposition 5.3 we have even $P = 0 \oplus 0 \oplus X_\kappa \oplus Y$ and $P^o = 0 \oplus X_{\kappa^o} \oplus 0 \oplus Y$.

Elements of the second type have the shape $(0, \kappa^o(e), e, 0)$ or $(0, e, \kappa(e), 0)$. Note that κ is a bijection on X and $\kappa^o = \kappa^{-1}$, so that $(0, \kappa^o(e), e, 0) = (0, f, \kappa(f), 0)$ with $f = \kappa^o(e)$. So the elements of the second type generate

$$D = \langle (0, e, \kappa(e), 0) \mid e \in X \rangle.$$

As we have seen before the conjugates of $P + P^o + D$ under N generate $P(\bar{\mathcal{S}})$. Since $[0 \oplus X \oplus X \oplus Y, N] \subseteq 0 \oplus 0 \oplus 0 \oplus Y$ we get $P(\bar{\mathcal{S}}) = P + P^o + D$. By the proof of Proposition 5.3 $P = 0 \oplus 0 \oplus X_\kappa \oplus Y \subseteq D \oplus (0 \oplus 0 \oplus 0 \oplus Y)$ and by symmetry $P^o \subseteq D \oplus (0 \oplus 0 \oplus 0 \oplus Y)$ holds too. Assertion (a) follows.

In particular $0 \oplus 0 \oplus 0 \oplus Y \subseteq P(\bar{\mathcal{S}}) \subseteq 0 \oplus X \oplus X \oplus Y = U_1$. Set $W = U_1/(0 \oplus 0 \oplus 0 \oplus Y)$ and identify W in the obvious way with $X \oplus X$, and $W_D = P(\bar{\mathcal{S}})/(0 \oplus 0 \oplus 0 \oplus Y)$ with $\langle (e, \kappa(e)) \mid e \in X \rangle$. The proof of Proposition 5.3 shows

$$\dim W_D = n + \dim X_\kappa.$$

So $\dim U_1 - \dim P(\bar{\mathcal{S}}) = \dim W - \dim W_D = n - \dim X_\kappa = \dim U(\mathcal{S}) - \dim P(\mathcal{S})$. Assertion (b) follows. □

6 Proof Theorem 1.3

A necessary step in the proof of Theorem 1.3 is the identification of quotients of the Huybrechts and Buratti-Del Fra DHOs. This will be achieved by using certain addition formulas of Taniguchi and Yoshiara (see [14] and [15]). For our applications it is convenient to present these in a slightly modified form.

6.1 Addition formulas

Definition 6.1. Let V, W be finite dimensional $\mathbb{F}_2$-spaces $\dim V \geq 2$. We call a mapping $f : V \times V \to W$ *symmetric* if for all $s, t \in V$

$$f(s,t) = f(t,s) \quad \text{and} \quad f(s,s) = 0$$

holds. We say that f is symmetric of *type (H)* if

$$f(s,t_1) + f(s,t_2) = f(s, s + t_1 + t_2)$$

holds for all $s, t_1, t_2 \in V$.

We say that f is symmetric of *type (D)* if there exist $0 \neq e_0 \in V$ such that

$$f(s,t_1) + f(s,t_2) = f(s, s + t_1 + t_2 + \alpha(s,t_1,t_2)e_0)$$

for all $s, t_1, t_2 \in V$. Here α is defined by $\alpha(x,y,z) = \xi(x+y) + \xi(y+z) + \xi(z+x)$ where $\xi : V \to \mathbb{F}_2$ is the characteristic function of $V' = V - \langle e_0 \rangle$.

Lemma 6.2. *Let f be symmetric of type (H). Let $\{e_1, \ldots, e_n\}$ be a basis of V. Set $\mathcal{B} = \{0, e_i \mid 1 \leq i \leq n\}$ with the obvious lexicographic order. Then for each $(s,t) \in V \times V$ there exist elements $a_{w,w'}(s,t) \in \mathbb{F}_2$, $w, w' \in \mathcal{B}$ depending only on the choice of the basis of V (but not on f) such that*

$$f(s,t) = \sum_{w,w' \in \mathcal{B},\, w < w'} a_{w,w'}(s,t) f(w,w').$$

Proof. For $s = 0$ the function f is additive in the second argument, i.e. $f(0,s) = \sum_i s_i f(0,e_i)$ for $s = \sum s_i e_i$. The assertion holds for $s = 0$.

We prove the claim by induction on $w(s)+w(t)$ where $w(s)$ is the Hamming weight of s. For $J \subseteq \{1,\dots,n\}$ and $j_0 \in J$ condition (H) shows

$$f(s, \sum_{j\in J} e_j) = f(s,0) + f(s,e_{j_0}) + f(s, \sum_{j\in J-\{j_0\}} e_j).$$

By the case $s=0$ and by symmetry the term $f(s,0)$ can be expressed as a linear combination of some $f(0,e_i)$. Induction on $w(s)+w(t)$ shows that the two other terms on the RHS can be expressed by elements of the form $f(0,e_i)$ and $f(e_i,e_j)$ (with $i<j$). The assertion follows. □

Let $\mathcal{H}_n$ the Huybrechts DHO of rank n In $U = U(\mathcal{H}_n) = X \oplus Y$, $X = \langle e_0, e_1, \dots, e_{n-1}\rangle$, $Y = \wedge^2(X)$ as described at the beginning of Subsection 3.2. Identifying X with $X \oplus 0$ we see $X \curlywedge X(s) = s$ and $X(s) \curlywedge X(t) = (s+t, s\wedge t)$. Define $h : X \times X \to U$ by $h(s,t) = X(s)\curlywedge X(t) = (s+t, s\wedge t)$. Then

$$h(s,t_1) + h(s,t_2) = (t_1+t_2, s\wedge(t_1+t_2)) = h(s, s+t_1+t_2).$$

So h is symmetric of type (H). The following characterization of quotients of Huybrechts DHOs is implicitly contained in [14] and [15] but for convenience we indicate a verification along the lines of [14, Prop. 1].

Proposition 6.3. *Let $\mathcal{S} = \{H(s) \mid s \in X\}$, $X = \langle e_0, e_1, \dots, e_{n-1}\rangle$, be a DHO of rank n over $\mathbb{F}_2$. For $s,t \in X$ set $b(s,t) = H(s)\curlywedge H(t)$. Suppose that b is symmetric of type (H). Then $\mathcal{S}$ is a quotient of $\mathcal{H}_n$.*

Proof. (Sketch) Set $\mathcal{B} = \{0, e_i \mid 1 \le i \le n\}$ with the obvious lexicographic order. Then the $b(w,w')$, $w,w' \in \mathcal{B}$, $w < w'$ form a basis of U. By Lemma 6.2 for $s,t \in V_0$ there exist $a_{w,w'}(s,t) \in \mathbb{F}_2$ such that for the mapping $f = b$ as well for the mapping $f = h$ the equation

$$f(s,t) = \sum_{w,w'\in\mathcal{B},\, w<w'} a_{w,w'}(s,t) f(w,w')$$

holds. So if we define a linear operator $\phi : U \to U(\mathcal{S})$ by $h(w,w')\phi = b(w,w')$, $w,w' \in \mathcal{B}$, we have $h(s,t)\phi = b(s,t)$. Hence $X(s)\phi = H(s)$ and ϕ maps $\mathcal{H}_n$ onto $\mathcal{S}$. □

Proposition 1 of [14] covers the analogous result for quotients of the Buratti-Del Fra DHOs

Proposition 6.4. *Let $\mathcal{S} = \{X(s) \mid s \in V\}$ be a DHO of rank n over $\mathbb{F}_2$ whose elements are indexed by the elements of V, $\dim V = n$. For $s,t \in V_0$ set $d(s,t) = X(s)\curlywedge X(t)$. Suppose that d is symmetric of type (D). Then $\mathcal{S}$ is a quotient of $\mathcal{D}_n$.*

These addition formulas can be derived in a fashion that is independent of the concrete model describing $\mathcal{H}_n$ or $\mathcal{D}_n$:

Lemma 6.5. *Let $\mathcal{H} = \mathcal{H}_n$ the Huybrechts DHO of rank n and U the ambient space. Let $Z \in \mathcal{H}$. For $W \in \mathcal{H}$ define $W = Z(w)$ if $W \curlywedge Z = w$ and define $h_Z : Z \times Z \to U$ by $h_Z(w, w') = Z(w) \curlywedge Z(w')$. Then h_Z is symmetric of type (H).*

Proof. Define the Huybrechts DHO with $X = X(0) = \langle e_0, e_1, \ldots, e_{n-1} \rangle$, as usual. Choose $Z = X$. Then we have seen that $h = h_X$ fulfills the assertion of the Lemma. Let ϕ be an automorphism of $\mathcal{H}$ such that $X\phi = Z$. Suppose $X \curlywedge W = w$ and $X \curlywedge W' = w'$. Then $Z \curlywedge W\phi = w\phi$ and $Z \curlywedge W'\phi = w'\phi$. Therefore $h_Z(w\phi, w'\phi) = (W \curlywedge W')\phi = h(w, w')\phi$. So h_Z is symmetric of type (H). Since the automorphism group acts transitively on $\mathcal{H}$ the assertion holds for an arbitrary choice of Z. □

Lemma 6.6. *Let $\mathcal{D} = \mathcal{D}_n$ the Buratti-Del Fra DHO of rank $n \geq 4$ and U the ambient space. Let $Z \in \mathcal{D}$. Let $P(\mathcal{D}) \cap Z = \langle e_0 \rangle$ and let $\xi : Z \to \mathbb{F}_2$ be the characteristic function of $Z - \langle e_0 \rangle$. For $W \in \mathcal{D}$ set $Z = Z(w)$ where $w = Z \curlywedge W + \xi(Z \curlywedge W + e_0)e_0$. Define $d_Z : Z \times Z \to U$ by $d_Z(w, w') = Z(w) \curlywedge Z(w')$. Then d_Z is symmetric of type (D).*

Proof. With the representation of $\mathcal{D}$ displayed in Subsection 3.1 we compute $d(s, t) = X(s) \curlywedge X(s) = (s + t + \xi(s + t)e_0, \overline{(s + t + \xi(s + t)e_0) \cdot s})$ which shows that $d = d_X$ (for $X = X(0)$) is symmetric of type (D). Let ϕ be an automorphism of $\mathcal{D}$ such that $X\phi = Z$. As $P(\mathcal{D})$ is invariant under ϕ we get $Z \cap P(\mathcal{D}) = \langle e_0\phi \rangle$ and $\xi_X(w) = \xi_Z(w\phi)$ (the subscript indicates the domain of the function ξ). This implies $d_Z(w\phi, w'\phi) = d(w, w')\phi$. So d_Z is symmetric of type (D). Since the automorphism group acts transitively on $\mathcal{D}$ the assertion holds for an arbitrary choice of Z. □

Remark 6.7. A covering map $f : \mathcal{S}' \to \mathcal{S}$ is injective on the elements contained in the members of $\mathcal{S}'$. So the assertion of Lemma 6.5 holds for every quotient of $\mathcal{H}_n$. By the same token the assertion of Lemma 6.6 holds for every quotient $\mathcal{S}$ of $\mathcal{D}_n$ with $\dim U(\mathcal{S}) - \dim P(\mathcal{S}) = n - 1$.

Lemma 6.8. *Let $n \geq 4$ and $\mathcal{S}$ be a quotient of $\mathcal{D}_n$ such that $\dim U(\mathcal{S}) - \dim P(\mathcal{S}) = n-1$. Let $W \subset X \in \mathcal{S}$ be a subspace of rank 3 such that $W \cap P(\mathcal{S}) = 0$. Then $\mathcal{S}[W]$ is not a subDHO.*

Proof. Assume first $\mathcal{S} = \mathcal{D}_n$ and choose $W = \langle e_1, e_2, e_3 \rangle \subset X(0)$. A direct computation shows that the claim holds in this special case. One knows that the stabilizer H of $X(0)$ in $\mathrm{Aut}(\mathcal{S})$ is isomorphic to the stabilizer of $\langle e_0 \rangle$ in $\mathrm{GL}(X(0))$ and H induces the natural action on $X(0)$. So H is transitive on the set of 3-spaces in $X(0)$ which do not contain $\langle e_0 \rangle$. So the assertion holds with respect all such 3-spaces. The members of $\mathcal{S}$ are conjugated under the automorphism group. So the assertion of the Lemma holds for $\mathcal{S} = \mathcal{D}_n$. But then the assertion holds even for proper quotients $\mathcal{S}$ of $\mathcal{D}_n$ provided $\dim U(\mathcal{S}) - \dim P(\mathcal{S}) = n - 1$. □

6.2 The case $n = 4$

The proof of Theorem 1.3 proceeds by induction on the rank n. The basis $n = 4$ is dealt with by computer calculation.

Let $\mathcal{S}$ be a DHO over $\mathbb{F}_2$ of rank 4 such that

$$\dim U(\mathcal{S}) \geq \dim P(\mathcal{S}) + 3.$$

The binary DHOs of rank 4 and whose ambient space has rank ≤ 8 are classified in [1]. For $\dim U(\mathcal{S}) = 7$, there is precisely one DHO with $\dim U(\mathcal{S}) - \dim P(\mathcal{S}) \geq 3$, namely the DHO with the ID-number 1 in [1, Table 1]. This DHO is a quotient of $\mathcal{D}_4$ (see [1, Subsec. 4.3]). For $\dim U(\mathcal{S}) = 8$, there are precisely two DHOs of with $\dim U(\mathcal{S}) - \dim P(\mathcal{S}) \geq 3$, namely Id-numbers 1 and 3 of [1, Table 1] which are a quotient of $\mathcal{H}_4$ and $\mathcal{D}_4$ respectively (see [1, Subsec. 4.3]).

So assume now $\dim U(\mathcal{S}) \geq 9$. Let Q be a subspace of codimension 1 in $U = U(\mathcal{S})$ which contains $P = P(\mathcal{S})$. By Theorem 4.6

$$\mathcal{S} = \mathcal{T}_1 \sqcup \mathcal{T}_2$$

with subDHOs $\mathcal{T}_i$ of rank 3. Binary DHOs of rank 3 have been classified by Del Fra [2]; there are three DHOs of this rank. One DHO has an ambient space of rank 5 while the other two DHOs have an ambient space of rank 6. By Proposition 4.8

$$\dim U \leq \dim U(\mathcal{T}_1) + 4 \leq 10.$$

By Theorem 4.6 each of the subDHOs $\mathcal{T}_i$ is the disjoint union of two subDHOs of rank 2. Inspecting the three DHOs from [2] we deduce $\mathcal{T}_i \simeq \mathcal{H}_3$, $i = 1, 2$.

We now describe an algorithm which enumerates possible DHOs $\mathcal{S}$ over $\mathbb{F}_2$ such that $\dim U(\mathcal{S}) = 9$ or $= 10$ and $\dim U(\mathcal{S}) - \dim P(\mathcal{S}) \geq 3$.

We start with a DHO $\mathcal{T}_1 \simeq \mathcal{H}_3$ such that the 6-space $U_1 = U(\mathcal{T}_1)$ is embedded in a space U of dimension 9 or 10 (the ambient space for the DHO $\mathcal{S}$ which we like to find). We pick $Z \in \mathcal{S} - \mathcal{S}(\mathcal{T}_1) = \mathcal{S}(\mathcal{T}_2)$. Then

$$U = U_1 + Z \tag{1}$$

and

$$Q = U_1 + Z_0, \tag{2}$$

is a hyperplane in U. Here $Z_0 \in \mathcal{T}_2$, with $Z_0 \subset Z$. Note that $U = U_1 \oplus Z$ for $\dim U = 10$ whereas $U_1 \cap Z = U_1 \cap Z_0$ has dimension 1 if $\dim U = 9$. Furthermore by Lemma 4.11

$$Z - Z_0 = \{X \curlywedge Z \mid X \in \mathcal{S}(\mathcal{T}_1)\}. \tag{3}$$

The algorithm has two steps. In Step 1 we compute all "starter-sets"

$$\mathcal{S}_0 = \mathcal{S}(\mathcal{T}_1) \cup \{Z\}$$

and in Step 2 we determine all "completions" of Step 1, i.e. all DHOs $\mathcal{S}$ in U such that $\mathcal{S}_0 \subset \mathcal{S}$.

STEP 1. Identify U with $\mathbb{F}_2^m$, $m = 9$ or $= 10$ and U_1 with $\langle e_1, \ldots, e_6 \rangle$ (the e_i's form the standard basis of $\mathbb{F}_2^m$). If $m = 10$ we take $Z = \langle e_7, \ldots, e_{10} \rangle$ and $Z_0 = \langle e_7, e_8, e_9 \rangle$. If $m = 9$ then $\mathcal{C} = U_1 - (\cup_{X_0 \in \mathcal{T}_1} X_0)$ has size 35 and we have 35 candidates for Z and Z_0, namely $Z = Z_w = \langle e_7, \ldots, e_9, w \rangle$, $w \in \mathcal{C}$ and we can take $Z_0 = \langle e_7, e_8, w \rangle$. Let $Z - Z_0 = \{z_1, \ldots, z_8\}$ and $\mathcal{T}_1 = \{X_1, \ldots, X_8\}$. For a permutation $\pi \in \mathrm{Sym}(8)$ set $X_i^\pi = \langle X_i, z_{\pi(i)} \rangle$. Then by Equation (3) any starter-set has the form $\mathcal{S}_\pi = \{X_1^\pi, \ldots, X_1^\pi, Z\}$. We test for a given π that (i) the spaces in $\mathcal{S}_\pi$ have dimension 4, (ii) that two spaces of $\mathcal{S}_\pi$ intersect in a 1-space and (iii) that three spaces of $\mathcal{S}_\pi$ intersect trivially.

STEP 2. Let $\mathcal{S}_0 = \{X_1, \ldots, X_9\}$ a starter-set from Step 1. Set $\bar{X}_i = X_i - \{X_i \curlywedge X_j \mid 1 \le j \le 9\}$ for $1 \le i \le 9$. Each of these sets has size 7. Let $\bar{X}_1 = \{x_1, \ldots, x_7\}$ and for $1 \le i \le 7$. Next we compute $\mathcal{E}_i$, $\mathcal{E}_i$ the set of 4-spaces S in U such that $S \curlywedge X_1 = x_i$ and that $\mathcal{S}_0 \cup \{S\}$ satisfies conditions (ii) and (iii) from above. Finally we would get the completions $\mathcal{S}$ of $\mathcal{S}_0$ by adding one element from each $\mathcal{E}_i$ to $\mathcal{S}_0$ and testing conditions (ii) and (iii). In practice it turns out that *either* one of the $\mathcal{E}_i$ is empty in which case there is no completion of $\mathcal{S}_0$ *or* all $\mathcal{E}_i$'s have size 1 in which case it turns out that $\mathcal{S} = \mathcal{S}_0 \cup \{S_i \in \mathcal{E}_i \mid 1 \le i \le 7\}$ *is* a DHO.

All DHOs found this way are either alternating or $\dim U(\mathcal{S}) - \dim P(\mathcal{S}) = n - 1$ and $b(s,t) = X(s) \curlywedge X(t)$ is symmetric of type (D), if the index of X is chosen analog to Lemma 6.6.

It is known (see [1]) that $\mathcal{H}_4$ and $\mathcal{D}_4$ each have precisely one quotient whose ambient space has rank m. So only these quotients are recovered by the computer search.

6.3 The case $n > 4$

Proof. (of Theorem 1.3) Let $n \ge 5$, $X \in \mathcal{S}$ and $P(\mathcal{S}) \cap X = \langle e_0 \rangle$. Denote by H the set of hyperplanes of X which contain e_0.

(1) Let $M \in H$. As $P(\mathcal{S}[M]) \subseteq P(\mathcal{S})$ we have for $X \in \mathcal{S}(M)$ that $\dim(X \cap H) \cap P(\mathcal{S}[M]) \le \dim X \cap P(\mathcal{S}) = 1$, so that by Lemma 2.1 $\dim U(\mathcal{S}[M]) - \dim P(\mathcal{S}[M]) \ge n - 2$. Moreover $\mathcal{S}[M]$ is a quotient of $\mathcal{H}_{n-1}$ or $\mathcal{D}_{n-1}$ by induction and Corollary 2.2.

(2) If $\mathcal{S}[K]$ is a subDHO for every 3-space K in X, then $\mathcal{S}$ is a quotient of $\mathcal{H}_n$:

For $W \in \mathcal{S}$ set $W = X(s)$ if $W \curlywedge X = s$. Define $h : X \times X \to U$ by $h(s,t) = X(s) \curlywedge X(t)$. By Proposition 6.3 it is enough to show that h is symmetric of type (H). Let $s, t_1, t_2 \in X$. Then there exists $Z \in H$ with $s, t_1, t_2 \in Z$. By (1) and Lemma 6.8 $\mathcal{S}[Z]$ is a quotient of $\mathcal{H}_{n-1}$. For $W \in \mathcal{S}[Z]$ define $W = Z(w)$ if $W \curlywedge Z = w$. Let $X(t) \in \mathcal{S}(Z)$ with $W \subseteq X(t)$. Then $w = t$ since $W \curlywedge Z =$

$X(t) \curlywedge X$. Hence the function $h_Z : Z \times Z \to U$ defined by $h_Z(s,t) = Z(s) \curlywedge Z(t)$ is the restriction of h to $Z \times Z$. On the other hand by Lemma 6.5 and Remark 6.7 h_Z is symmetric of type (H). The claim follows.

We assume from now on that (*) *there exists a* 3*-space* K *in* X *such that* $\mathcal{S}[K]$ *is not a subDHO.*

(3) For every $M \in H$ the subDHO $\mathcal{S}[M]$ is a quotient of $\mathcal{D}_{n-1}$ and $\dim U(\mathcal{S}[M]) - \dim P(\mathcal{S}[M]) = n-2$.

Note that $e_0 \notin K$ by Theorem 4.6. Every subDHO $\mathcal{S}[M]$ with $K \subset M$ can not be a quotient of $\mathcal{H}_{n-1}$, i.e. such a DHO is a quotient of $\mathcal{D}_{n-1}$. But then for *all* 3-spaces $e_0 \notin K' \subset M$ the set $\mathcal{S}[K']$ is not a subDHO (Lemma 6.8). But if $N \in H$ then $N \cap M$ contains such a 3-space. The assertion follows.

(4) Assume (*). Then $\mathcal{S}$ is a quotient of $\mathcal{D}_n$.

For $W \in \mathcal{S}$ set $W = X(s)$ if $W \curlywedge X = s + \xi(X \curlywedge W + e_0)e_0$. Define $d : X \times X \to U$ by $d(s,t) = X(s) \curlywedge X(t)$. By Proposition 6.4 it is enough to show that d is symmetric of type (D). Let $s, t_1, t_2 \in X$. Then there exists $Z \in H$ with $s, t_1, t_2 \in Z$. By (3) $\mathcal{S}[Z]$ is a quotient of $\mathcal{D}_{n-1}$. For $W \in \mathcal{S}[Z]$ define $W = Z(w)$ if $W \curlywedge Z = w + \xi(Z \curlywedge W + e_0)e_0$. Let $X(t) \in \mathcal{S}(Z)$ with $W \subseteq X(t)$. Then $s = t$ since $W \curlywedge Z = X(t) \curlywedge X$. Hence the function $d_Z : Z \times Z \to U$ defined by $d_Z(s,t) = Z(s) \curlywedge Z(t)$ is the restriction of d to $Z \times Z$. On the other hand by Lemma 6.6 and Remark 6.7 d_Z is symmetric of type (D). The claim follows. □

7 Further examples and computations

Recall the representation of DHOs by *DHO sets* (see [1]). The examples of DHOs with a proper radical discussed so far are all quotients of extensions of bilinear DHOs. In particular any quotient of a Huybrechts DHO or of a Buratti-Del Fra DHO is of this "standard type". We present in this section simply connected DHOs with a proper radical which are not of this "standard type". Five examples have rank 4 (they are a byproduct of [1]), more than 250 have rank 5 and one has rank 6. These DHOs are provided in [8].

Example 7.1. The following five, simply connected examples of rank 4 are all of the form $\mathcal{S} = \mathcal{S}_1 \sqcup \mathcal{S}_2$, $\mathcal{S}_i \simeq \mathcal{H}_3$ is a Huybrechts DHO of rank 3. They occur in the following way as entrances in [1, Tab.1, Tab. 2]:

Rank 4				
Table	ID	$\dim U(\mathcal{S})$	$\dim P(\mathcal{S})$	bilinear
1	6	7	6	yes
1	10	7	6	no
1	24	7	6	no
2	2	8	6	yes
2	4	8	6	no

Example 7.2. The construction of the examples of rank 5 is based on Proposition 4.12. Suppose $\mathcal{S}$ is a bilinear DHO of rank 5 which contains a bilinear subDHO $\mathcal{T}$ of rank 4 which induced by a hyperplane H containing the centralizer of the standard translation group in $U(\mathcal{S})$. We call such a DHO a *prolongation of a rank 4 DHO*. A simple search procedure for prolongations is described in [8]. Excluding extensions we found 268 prolongations of bilinear DHOs of rank 4.

We also computed the universal covers of these prolongations; a simple algorithm for the computation of universal covers of split DHOs is described in [6, Sec. 2.5]. We obtained in this way 32 simply connected DHOs with a proper radical; 12 of these simply connected DHOs are extensions of bilinear rank 4 DHOs.

Example 7.3. The example $\mathcal{S}$ of rank 6 is bilinear, simply connected $\dim U(\mathcal{S}) = 11$ and $\dim P(\mathcal{S}) = 10$. Let $\mathcal{T}$ in $V = \mathbb{F}_2^{12}$ be the semifield spread of the semifield which occurs under number XII in [11]. Let $Z \in \mathcal{T}$ be the component which is the centralizer of the elation group. Then there exists a 1-space $Q \subseteq Z \subseteq V$ such that the DHO is the quotient $\mathcal{S} = \mathcal{T}/Q = \{(T+Q)/Q \mid T \in \mathcal{T} - \{Z\}\}$ in the ambient space $U(\mathcal{S}) = V/Q$. One has $\mathcal{S} = \mathcal{S}_1 \sqcup \mathcal{S}_2$ where $\mathcal{S}_1 \simeq \mathcal{S}_2$ is a bilinear symplectic DHO of rank 5 described in [3, Ex. 3.14]. The automorphism group of $\mathcal{S}$ has the form $\mathrm{Aut}(\mathcal{S}) = T \cdot H$ where T is the (normal) translation group and $H \simeq \mathrm{Sym}(3)$ is the stabilizer in the automorphism group of $X(0)$.

Remark 7.4. (a) For a given rank the proportion of binary DHOs with a proper radical seems to be quite small in comparison to all isomorphism types. For instance we could not find a DHO of rank 5 which is constructed as a quotient of a spread set of a translation plane of order 32 and from the 80 semifield planes of order 64 only one case produced an example (Ex. 7.3).

(b) The examples of this section indicate that "non-standard type" DHOs with a proper radical should exist for arbitrary rank. Yoshiara [17] gives conditions how to construct from a DHO $\mathcal{T}$ of rank n a DHO $\mathcal{S}$ of rank $n+1$ such that (1) $\mathcal{T}$ occurs in $\mathcal{S}$ as a hyperplane-induced subDHO and that (2) $\dim U(\mathcal{S}) = n + 1 + \dim U(\mathcal{T})$. Bilinear DHOs $\mathcal{T}$ lead to the extension construction. For non-bilinear DHOs $\mathcal{T}$ there is no easy criterion guaranteeing the existence of $\mathcal{S}$. More general: The construction of a *series* of "non-standard type" DHOs with a proper radical is an open problem.

Remark 7.5. Quotients of binary rank 5 DHOs with maximal ambient spaces.

For a binary DHO $\mathcal{S}$ of rank n an upper bound for the dimension of the ambient space is $\binom{n+1}{2} + 2$ and it is conjectured that even $\dim U(\mathcal{S}) \leq \binom{n+1}{2}$ holds (see [16]). Presently for $n \geq 4$ precisely 4 binary DHOs of rank n with an ambient space of dimension $\binom{n+1}{2}$ are known, namely $\mathcal{H}_n$, $\mathcal{D}_n$, the Veronesean DHO $\mathcal{V}_n$ and the Taniguchi DHO $\mathcal{T}_n$. We computed all quotients of $\mathcal{H}_5$, $\mathcal{D}_5$, $\mathcal{V}_5$ and $\mathcal{T}_5$ (see [8]). These data show for instance:

(a) There are quotients of $\mathcal{H}_5$ and $\mathcal{D}_5$ which are not bilinear.

(b) There are quotients $\mathcal{S}$ of $\mathcal{D}_5$ with $U(\mathcal{S}) = P(\mathcal{S})$ for $\dim U(\mathcal{S}) = 10$ or $= 11$.

Up to this date only DHOs of split type were known (see in particular Yoshiara [17], [18] for thorough investigations of the splitness of DHOs). The most surprising discovery of our computations is:

$\mathcal{V}_5$ and $\mathcal{T}_5$ have quotients of non-split type.

References

[1] A. Betten, U. Dempwolff, A. Wassermann: On dual hyperovals of rank 4 over $\mathbb{F}_2$, J. Geom. 108(2017), 75-98.

[2] A. Del Fra: On d-dimensional dual hyperovals, Geom. Dedicata 79(2000), 157-178.

[3] U. Dempwolff: Dimensional doubly dual hyperovals and bent functions, Innov. Incidence Geom., 13 (2013), 146-178.

[4] U. Dempwolff: Universal covers of dimensional dual hyperovals, Discrete Math., 338(2015), 633-636.

[5] U. Dempwolff: The non-solvable doubly transitive dimensional dual hyperovals, Adv. Geom. 18(2018), 1-4.

[6] U. Dempwolff: Doubly transitive dimensional dual hyperovals: universal covers and non-bilinear examples, Adv. Geom. 19(2019), 359-379.

[7] U. Dempwolff, Y. Edel: Dimensional dual hyperovals and APN functions with translation groups, J. Algebraic Combin., 39(2014), 457-496.

[8] U. Dempwolff, Y. Edel: The webpage associated with the current paper, containing the additional files. `http://www.mathi.uni-heidelberg.de/~yves/Papers/radical.html`

[9] U. Dempwolff, W. Kantor: Orthogonal dual hyperovals, symplectic spreads and orthogonal spreads, J. Algebr Comb 41(2015), 83-108.

[10] Y. Edel: On quadratic APN functions and dimensional dual hyperovals, Designs, Codes and Cryptography, 57(2010), 35-44.

[11] I. Rua, E. Combarro, J. Ranilla: Classification of semifields of order 64, J. Algebra, 322(2009), 4011-4029.

[12] H. Taniguchi: Some examples of simply connected dual hyperovals, Finite Fields Appl. 22(2013), 45-50.

[13] H. Taniguchi, S. Yoshiara: On dimensional dual hyperovals $\mathcal{S}^{d+1}_{\sigma,\phi}$, Innov. Incidence Geom. 1(2005), 197-219.

[14] H. Taniguchi, S. Yoshiara: A new construction of the d-dimensional Buratti-Del Fra dual hyperoval, European J. Combin. 33(2012), 1030-1042.

[15] H. Taniguchi, S. Yoshiara: A unified description of four simply connected dimensional dual hyperovals, European J. Combin. 36(2014), 143-150.

[16] S. Yoshiara: Dimensional dual arcs–a survey, pp. 247–266 in: Finite geometries, groups, and computation (Eds. A. Hulpke et al.), de Gruyter, Berlin 2006.

[17] S. Yoshiara: Disjoint unions of dimensional dual hyperoval and its splitness, Innov. Incidence Geom. 14(2015), 43-76.

[18] S. Yoshiara: Splitness of the Veronesean and the Taniguchi dual hyperovals, Discrete Math. 342(2019), 844-854.