The classification of the largest caps in $AG(5,3)$

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Abstract

We prove that 45 is the size of the largest caps in $AG(5,3)$, and such a 45-cap is always obtained from the 56-cap in $PG(5,3)$ by deleting an 11-hyperplane.

1 Introduction

A *k-cap K* in $AG(n,q)$, respectively in $PG(n,q)$, is a set of k points in $AG(n,q)$, respectively in $PG(n,q)$, such that no three points are collinear.

A k-cap of $AG(n,q)$, respectively $PG(n,q)$, is called *complete* when it cannot be extended to a larger cap of $AG(n,q)$, respectively $PG(n,q)$.

The main problem in the theory of caps is to find the maximal size of a cap in $AG(n,q)$ or $PG(n,q)$.

Presently, only the following exact values are known. In $AG(2,q)$ and $PG(2,q)$, q odd, there are at most $(q+1)$-caps [3]. In $AG(2,q)$ and $PG(2,q)$, q even, there are at most $(q+2)$-caps [3]. In $AG(3,q)$, $q>2$, the maximal size of a cap is q^2, and in $PG(3,q)$, $q>2$, the maximal size of a cap is q^2+1 [3, 17]. And in $AG(n,2)$ and in $PG(n,2)$, the maximal size of a cap is 2^n [3].

In some cases, a complete characterization is known. Namely, in $AG(2,q)$ and in $PG(2,q)$, q odd, every $(q+1)$-cap is a conic [18, 19]. In $AG(2,q)$ and $PG(2,q)$, q even, $q \geq 16$, distinct types of $(q+2)$-caps exist; see [12] for a list of the known infinite classes of $(q+2)$-caps. In $PG(3,q)$, q odd, every (q^2+1)-cap is an elliptic quadric and in $AG(3,q)$, q odd, every q^2-cap is an elliptic quadric minus one point [1, 15]. In $PG(3,q)$, $q=2^h$, h odd, $h \geq 3$, next to the elliptic quadric, at least one other type of (q^2+1)-cap exists, called the *Tits ovoid* [21]. In $AG(3,q)$, q even, $q>2$, every q^2-cap is obtained by deleting one point from a (q^2+1)-cap in $PG(3,q)$. In $PG(n,2)$, every 2^n-cap is the complement of a hyperplane [20].

Apart from these results which are either valid for arbitrary q or arbitrary dimension n, only some other sporadic results are known. Namely, the maximal size of a cap in $AG(4,3)$ and in $PG(4,3)$ is 20 [16]. The maximal size of a cap in $PG(5,3)$ is 56 [7]. And the maximal size of a cap in $PG(4,4)$ is 41 [6].

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Regarding the characterizations, exactly one type of 20-cap exists in $AG(4, 3)$ and exactly 9 types of 20-caps exist in $PG(4, 3)$ [9]. The 56-cap in $PG(5, 3)$ is projectively unique [8]. And there are at least 2 distinct types of 41-caps in $PG(4, 4)$ [6].

In the other cases, only upper bounds on the sizes of caps in $AG(n, q)$ and $PG(n, q)$ are known. We refer to [12] for a list of the known results. We also wish to state the following result of Bierbrauer and Edel [2] which improves the Meshulam upper bound on the size of caps in $AG(n, q)$, q odd [14].

Theorem 1.1 *Let $Q = q^h, q > 2$ and $n \geq 4$. Then the size of a cap in $AG(n, Q)$ is upper bounded by*

$$\frac{(nh + 1)Q^n}{(nh)^2}.$$

We focus in this article on the maximal size of a cap in $AG(5, 3)$ and its relation to the 56-cap in $PG(5, 3)$. This latter 56-cap in $PG(5, 3)$, called the *Hill cap*, intersects a hyperplane of $PG(5, 3)$ in either 20 or 11 points.

Hence, defining $AG(5, 3)$ to be $PG(5, 3)$ minus an 11-hyperplane of this 56-cap, we obtain that there exists a 45-cap in $AG(5, 3)$.

No larger caps are known in $AG(5, 3)$.

Presently, the best upper bound on the size of a cap in $AG(5, 3)$ is by Bruen, Haddad and Wehlau [4] who proved that the size of a cap in $AG(5, 3)$ is at most 48.

We prove in this article the following theorem.

Theorem 1.2 *The maximal size of a cap in $AG(5, 3)$ is equal to 45, and every 45-cap in $AG(5, 3)$ is obtained by deleting an 11-hyperplane from a 56-cap in $PG(5, 3)$.*

Moreover, there is a unique type of 45-caps in $AG(5, 3)$.

2 Preliminary results

The following result was already mentioned in the introduction, but we repeat it since it is frequently used in this article.

Lemma 2.1 *The largest cap in $AG(3, 3)$ is a 9-cap obtained by deleting a 1-hyperplane from an elliptic quadric in $PG(3, 3)$.*

Proof: See for instance [11, p. 104]. □

A set K of n points of $PG(k - 1, q)$ is called an $(n, m; k - 1, q)$*-set*, or (n, m)*-set* for short, if K meets every hyperplane in at most m points. The existence of a projective $[n, k, d]_q$ code of full length (no coordinate position identically zero) is equivalent to the existence of an $(n, n - d)$-set in $PG(k - 1, q)$. For a detailed investigation of this correspondence, we refer to [5] and [13].

Given an $(n, n-d)$-set K in $PG(k-1,q)$, we denote by n_i the number of hyperplanes H in $PG(k-1,q)$ with $|K \cap H| = i$. We call the sequence of integers $\{n_i\}_{i\geq 0}$ the *spectrum* of K. Simple counting arguments yield the following identities for n-caps in $PG(k-1,q)$:

$$\begin{array}{lcl} \sum_{i\geq 0} n_i & = & \frac{q^k-1}{q-1} \\ \sum_{i\geq 0} i n_i & = & n\frac{q^{k-1}-1}{q-1} \\ \sum_{i\geq 0} i(i-1) n_i & = & n(n-1)\frac{q^{k-2}-1}{q-1} \\ \sum_{i\geq 0} i(i-1)(i-2) n_i & = & n(n-1)(n-2)\frac{q^{k-3}-1}{q-1}. \end{array} \quad (1)$$

Let $\mathcal{P}$ be the set of points of $PG(k-1,q)$ and let π and σ be disjoint flats of dimensions i and j, respectively, with $i + j = k-2$. We define the *projection* $\varphi_{\pi,\sigma}$ *from* π *onto* σ by

$$\varphi_{\pi,\sigma} \colon \mathcal{P} \setminus \pi \to \sigma : Q \mapsto \sigma \cap \langle \pi, Q \rangle, \quad (2)$$

where $\langle \pi, Q \rangle$ is the $(i+1)$-dimensional flat generated by π and Q. Let us note that $\varphi_{\pi,\sigma}$ maps flats of dimension $(i+s)$ containing π into $(s-1)$-dimensional flats in σ. Given an (n,m)-set K and a set of points $\mathcal{F} \subset \sigma$, we define $\mu(\mathcal{F}) = |\{P \in K \mid \varphi(P) \in \mathcal{F}\}|$. If $\mathcal{F}$ is a k'-dimensional flat in σ and $|K \cap \pi| = w$ then $\mu(\mathcal{F}) \leq \gamma_{k'+i+1} - w$, where $\gamma_{k'+i+1} = |PG(k'+i+1,q)|$. Let l be a line in σ incident with the points $P_0, P_1, \ldots, P_q$. We call the $(q+1)$-tuple $(\mu(P_0), \mu(P_1), \ldots, \mu(P_q))$ the *type* of l, and we call $\mu(P_i)$ the *weight* of the point P_i.

We call the 1-, 2-, 3- and $(k-2)$-dimensional flats *lines*, *planes*, *solids*, and *hyperplanes*, respectively. If K is an (n,m)-set, then an *i-line* (with respect to K) is a line l with $|K \cap l| = i$; *i-planes, i-solids*, and *i-hyperplanes* are defined in a similar way.

By [22], there are exactly 7 different $(18,8)$-sets in $PG(4,3)$. Each $(18,8)$-set is uniquely extendable to a $(20,8)$-set. There are exactly 2 types of $(18,8)$-sets, which are also affine (this corresponds to the fact that two of the seven $[18,5,10]$-codes have maximum weight 18). Also a $(9,5)$-set in $AG(4,3)$ is uniquely extendable to an $(11,5)$-set in $PG(4,3)$; this corresponds to the dual *Golay code*. We also remark that a solid in $PG(4,3)$ intersects an $(11,5)$-set in 5 or 2 points.

In $PG(4,3)$, we will project an affine (18,8)-set, respectively an affine (9,5)-set from an empty plane π onto some line l disjoint from π. The next table lists the possible types of the lines which are images of such sets under this projection. It is assumed that π is contained in a 0-solid δ. The column "# of π's" gives the number of choices for the empty plane π in δ, for which we get the particular type for the line l.

Table 1. The types of the images of (18,8)- and (9,5)-sets in $PG(4,3)$ under a projection from an empty plane contained in a 0-solid.

		Type	# of π's
$(18,8)$	(A)	(8,8,2,0)	9
	(B)	(8,5,5,0)	9
	(C)	(7,7,4,0)	18
	(D)	(6,6,6,0)	4
$(9,5)$	(E)	(5,2,2,0)	18
	(F)	(4,4,1,0)	18
	(G)	(3,3,3,0)	4

Remark 2.2 Let π be the plane at infinity, from which we project. Types (A), (B) (respectively (E)) correspond to the case that π contains none of the two points which extend the $(18,8)$-set (respectively the $(9,5)$-set). Type (C) (respectively (F)) corresponds to the case that π contains one of the two points which extend the $(18,8)$-set (respectively the $(9,5)$-set). Type (D) (respectively (G)) corresponds to the case that π contains the two points which extend the $(18,8)$-set (respectively the $(9,5)$-set).

3 The size of a largest cap in $AG(5,3)$

Theorem 3.1 *The largest size of an n-cap in $AG(5,3)$, with at most 18 points in every hyperplane, is 45.*

Moreover, every 45-cap in $AG(5,3)$ contains at least one 18-, 19-, or 20-hyperplane.

Proof: This follows from [2, Theorem 5]. More precisely, the size of a cap in $AG(k,q)$, having at most a c-hyperplane, is at most $q^k(1+cq)/(q^k+cq)$.

An elementary counting argument shows that there is at least one 18-, 19- or 20-hyperplane. □

Lemma 3.2 *Let K be a 45-cap in $AG(5,3)$. Let $P(i)=(i-r_1)(i-r_2)(i-r_3)$, for some constants r_1,r_2,r_3. Then we have the following equality:*

$$\begin{aligned}\sum_i P(i)n_i &= 1106820+(3-r_1-r_2-r_3)79200 \\ &\quad +(r_1r_2+r_2r_3+r_1r_3+1-r_1-r_2-r_3)5445-363r_1r_2r_3. \qquad (3)\end{aligned}$$

Proof: We have the following equalities:

$$\begin{aligned}\sum_i n_i &= 363, \\ \sum_i in_i &= 45\times 121, \\ \sum_i \binom{i}{2}n_i &= \binom{45}{2}40, \\ \sum_i \binom{i}{3}n_i &= \binom{45}{3}13.\end{aligned}$$

Equation (3) follows from

$$P(i) = 6\binom{i}{3} + (6 - 2r_1 - 2r_2 - 2r_3)\binom{i}{2} + (r_1r_2 + r_2r_3 + r_1r_3 + 1 - r_1 - r_2 - r_3)i - r_1r_2r_3.$$

□

Lemma 3.3 *Assume there exists a* 45*-cap* K *in* $AG(5,3)$*, for which there exists a hyperplane which intersects in more than* 18 *points. Then we can always find either a* 5*-,* 6*-, or* 7*-hyperplane parallel to a* 20*-hyperplane or a* 7*- or* 8*-hyperplane parallel to a* 19*-hyperplane.*

Proof: Let $P(i) = (i-11)(i-15)(i-16)$, then equation (3) gives $\sum_i P(i)n_i = 0$. Assume that there are no 20-hyperplanes, but there is a 19-hyperplane. Suppose there are no 7-hyperplanes. An 8-hyperplane and its parallel 18- and 19-hyperplane contribute -30 to (3) (using $(r_1, r_2, r_3) = (11, 15, 16)$), while a 9-hyperplane and two parallel 18-hyperplanes, and three parallel 15-hyperplanes contribute zero to (3). All other triples of parallel hyperplanes contribute a positive number to (3). Hence, if there is no 8-hyperplane, there are only 9-, 15- or 18-hyperplanes; but this contradicts the assumption that there is a 19-hyperplane. So, parallel to some 19-hyperplane, there is a 7- or a 8-hyperplane.

Suppose there is a 20-hyperplane. A 5-hyperplane or a 6-hyperplane is always parallel to a 20-hyperplane. A 7-hyperplane is parallel to a 19- or a 20-hyperplane. So assume $n_5 = n_6 = n_7 = 0$. As a 20-hyperplane and its two parallel hyperplanes always induce a positive contribution to (3) for $(r_1, r_2, r_3) = (11, 15, 16)$, there must be a negative contribution. As above, this is only possible for a parallel 8-, 18-, 19-hyperplane triple. □

Lemma 3.4 *There is no* 45*-cap in* $AG(5,3)$ *for which there exists a hyperplane intersecting in more than* 18 *points.*

Proof: ¿From [9], we know that there is a unique 20-cap in $AG(4,3)$ and a computer search for all 19-caps in $AG(4,3)$ showed that there is a unique 19-cap.

Using a similar computer search as in [6], we eliminated all cases occurring in Lemma 3.3. □

4 The classification of the 45-caps in $AG(5,3)$

Remark 4.1 There exist 45-caps in $AG(5,3)$, since the Hill-cap is a 56-cap in $PG(5,3)$ which contains an 11-hyperplane [8]. Deleting such an 11-hyperplane yields a 45-cap in $AG(5,3)$.
We are going to prove that every 45-cap in $AG(5,3)$ is obtained in that way.

¿From the preceding lemma, we know that there are at most 18-hyperplanes.

Lemma 4.2 *Let* K *be a* 45*-cap in* $AG(5,3)$*. Then every hyperplane intersects* K *in either* 9*,* 15 *or in* 18 *points, and the spectrum of* K *is* $(n_9, n_{15}, n_{18}) = (55, 198, 110)$*.*

Proof: Let $P(i) = (i-11)(i-15)(i-16)$, then equation (3) gives $\sum_i P(i)n_i = 0$. We count the contribution of parallel hyperplane triples to this sum. Only a 9-hyperplane parallel to two 18-hyperplanes and three parallel 15-hyperplanes give a zero contribution. All other contributions are strictly positive. Hence we have only 9-, 15- and 18-hyperplanes, and $n_{18} = 2n_9$.

Take $P(i) = (i-11)(i-16)(i-16)$, then equation (3) gives $-98n_9 + 4n_{15} + 28n_{18} = -1518$. Using $n_9 + n_{15} + n_{18} = 363$, we get the spectrum of K. □

Definition 4.3 ([4]) *We define for a k-cap K in $AG(5,3)$, an* intersection square *in the following way. Take a hyperplane K_1 and its parallel hyperplanes K_2 and K_3. Take another hyperplane H_1 together with its parallel hyperplanes H_2 and H_3. An intersection square determined by H_1 and K_1 is the 3×3 matrix $[l_{ij}]$, where $l_{ij} = |L_{ij} \cap K|$, with $L_{ij} = H_i \cap K_j$.*

Remark 4.4 We remark that a cap has in general several intersection squares. The hyperplanes $L_{12} \cup L_{21} \cup L_{33}$, $L_{13} \cup L_{22} \cup L_{31}$ and $L_{23} \cup L_{32} \cup L_{11}$ form a parallel hyperplane triple, and also $L_{11} \cup L_{22} \cup L_{33}$, $L_{21} \cup L_{32} \cup L_{13}$ and $L_{31} \cup L_{12} \cup L_{23}$ form a parallel hyperplane triple. Actually, these four parallel hyperplane triples correspond to the parallel hyperplane triples going through the four solids containing the plane at infinity, contained in $H_1 \cap K_1$.

Lemma 4.5 *If K is a 45-cap in $AG(5,3)$ containing a 9-solid, then K has an intersection square of the form*

$$\begin{matrix} 9 & 0 & 9 \\ 3 & 3 & 3 \\ 6 & 6 & 6 \end{matrix} \tag{4}$$

Proof: Put $l_{11} = 9$. By Lemma 4.2, a 9-solid is contained in four 18-hyperplanes. Hence, $l_{11} + l_{12} + l_{13} = l_{11} + l_{22} + l_{33} = l_{11} + l_{21} + l_{31} = l_{11} + l_{23} + l_{32} = 18$. Lemma 4.2 implies that an 18-hyperplane is parallel to a 9-hyperplane and an 18-hyperplane. Using a computer program, we looked for all possibilities to complete our intersection square. Up to equivalence, the only possibility is the intersection square (4). □

Lemma 4.6 *If K is a 45-cap in $AG(5,3)$, then, up to equivalence, the possible intersection squares are*

$$\begin{matrix} 9 & 0 & 9 \\ 3 & 3 & 3 \\ 6 & 6 & 6 \end{matrix}, \quad \begin{matrix} 2 & 8 & 8 \\ 8 & 5 & 5 \\ 5 & 2 & 2 \end{matrix}, \quad \begin{matrix} 3 & 3 & 3 \\ 6 & 6 & 6 \\ 6 & 6 & 6 \end{matrix}, \quad \begin{matrix} 1 & 7 & 7 \\ 7 & 4 & 4 \\ 7 & 4 & 4 \end{matrix}, \quad \begin{matrix} 5 & 5 & 5 \\ 5 & 5 & 5 \\ 5 & 5 & 5 \end{matrix}.$$

Proof: In this argument, we heavily rely on Lemma 4.2, stating that there are only 9-, 15-, and 18-hyperplanes. Let S be a a-solid and consider the intersection square determined by S. Let n'_i denote the number of i-hyperplanes in the intersection square which contain S. Then clearly $n'_9 + n'_{15} + n'_{18} = 4$. Also summing the other eight entries in the intersection square we find $(9-a)n'_9+(15-a)n'_{15}+(18-a)n'_{18}=45-a$. Eliminating n'_9 from these two equations we find that $2n'_{15}+3n'_{18}=3+a$.

A 0-solid has to be contained in an 18-hyperplane and in three 9-hyperplanes. The solids in the 18-hyperplane, parallel to the 0-solid, have to be 9-solids. Hence we are in the case of Lemma 4.5 and, up to equivalence, the only possible intersection square containing a 0-solid is (4).

Assume $l_{11}=1$, we try to complete this to a valid intersection square. By the reasoning above, we can assume that there are no 0-solids in the intersection square. Also, we may assume that we have no 9-solids in the intersection square (Lemma 4.5). A 1-solid has to be contained in two 15-hyperplanes and in two 9-hyperplanes. Hence, we may assume that $l_{11}+l_{12}+l_{13}=l_{11}+l_{21}+l_{31}=15$. If we put $(l_{12},l_{13})=(6,8)$, then we cannot complete this to a valid intersection square, taking into consideration Lemma 4.2. So assume $l_{12}=l_{13}=l_{21}=l_{31}=7$. A 7-solid has to be contained in two 15-hyperplanes and in two 18-hyperplanes. Using $l_{12}+l_{21}+l_{33}\in\{15,18\}$ and $l_{11}+l_{22}+l_{33}=9$, we are reduced to two possibilities, namely $\begin{matrix}1&7&7\\7&7&\\7&&1\end{matrix}$ or $\begin{matrix}1&7&7\\7&4&\\7&&4\end{matrix}$. In the former case, the 7-solid corresponding to L_{12} lies already in two 15-hyperplanes $L_{12}\cup L_{21}\cup L_{33}$ and $L_{11}\cup L_{12}\cup L_{13}$; hence the other hyperplanes containing this solid have to be 18-hyperplanes. So $l_{32}=l_{23}=4$. But then the hyperplane $L_{31}\cup L_{32}\cup L_{33}$ is a 12-hyperplane; contradicting Lemma 4.2. In the latter case, $l_{12}+l_{22}+l_{32}$ has to be 15 or 18. So, l_{32} is 4 or 7. If $l_{32}=7$, then $l_{23}=1$ and $l_{13}+l_{23}+l_{33}=12$. This contradicts Lemma 4.2. Hence $l_{32}=4$ and $l_{23}=4$.

By a similar reasoning, we determined, up to equivalence, the possible intersection squares containing a 2-solid or a 3-solid: $\begin{matrix}2&8&8\\8&5&5\\5&2&2\end{matrix}$, $\begin{matrix}3&3&3\\6&6&6\\6&6&6\end{matrix}$, under the assumption that there are no 0- or 1-solids, and, in the latter case, 2-solids.

Assume $l_{11}=4$. Assume that we have no solids intersecting in less than 4 points. A 4-solid is contained in a 9-hyperplane, two 15-hyperplanes and an 18-hyperplane. But, since every entry in our intersection square is at least 4, we cannot obtain a 9-hyperplane.

If we assume that there are no solids sharing less than 5 points with the 45-cap, the only possible intersection square containing a 5-solid is $\begin{matrix}5&5&5\\5&5&5\\5&5&5\end{matrix}$.

A solid which intersects the cap in more than 5 points, has to be parallel with a solid intersecting in at most 5 points. □

4.1 Suppose there is no solid intersecting in 9 points

If there are no 9-solids, Lemma 4.6 yields that the 9 points of K lying in a 9-hyperplane form a $(9,5)$-set. Clearly, a solid in H_0, the empty hyperplane, is contained in either one 9- and two 18-hyperplanes or in three 15-hyperplanes. Since $(n_9,n_{15},n_{18})=(55,198,110)$ (Lemma 4.2), the first possibility occurs for 55 solids; the second occurs for the remaining 66 solids in H_0.

Let H_1 be a 9-hyperplane and let $\delta = H_0 \cap H_1$. Denote by P_δ and Q_δ the two points in δ which extend $K \cap H_1$ to an (11,5)-set in H_1 (Section 2). Now define L to be the union of all $\{P_\delta, Q_\delta\}$, where δ runs over all solids in H_0 contained in 9-hyperplanes. We are going to prove that $K \cup L$ is a (56,20)-set, and by [10], such a set is always a cap. Let H_2 and H_3 be the other hyperplanes through $\delta = H_0 \cap H_1$.

We will consider the hyperplanes in $PG(5,3)$, hence in the type of a hyperplane, we will have a fourth entry corresponding to H_0.

Lemma 4.7 *The sets* $(K \cap H_2) \cup \{P_\delta, Q_\delta\}$ *and* $(K \cap H_3) \cup \{P_\delta, Q_\delta\}$ *are (equivalent to)* $(20,8)$*-sets in* $PG(4,3)$*.*

Proof: Let P_1 and Q_1 be the points which extend $K \cap H_2$ to a (20,8)-set. Consider a plane π in δ which contains P_δ and Q_δ. ¿From Remark 2.2 and Table 1 (G), we know that $\varphi(H_1)$ is of type $(3,3,3,0)$. ¿From the third intersection square of Lemma 4.6, we obtain that for the 18-hyperplane H_2 parallel to H_1, we have that $\varphi(H_2)$ is of type $(6,6,6,0)$. Now it follows from Remark 2.2 and Table 1 (D) that π also contains the points P_1 and Q_1. Letting π vary in δ, we have that P_δ, Q_δ, P_1 and Q_1 are collinear.

Assume $\{P_\delta, Q_\delta\} \neq \{P_1, Q_1\}$. Let further $P_\delta \notin \{P_1, Q_1\}$ and consider a plane π in H_0 containing P_δ and none of the remaining three points. A similar reasoning as above shows that, if φ is the projection from π, the line $\varphi(H_1)$ is of type (4,4,1,0) while $\varphi(H_2)$ is of type (8,8,2,0) or (8,5,5,0), by Table 1 and Remark 2.2. This contradicts Lemma 4.6 since $(4,4,1)$ appears in the fourth intersection square while $(8,8,2)$ and $(8,5,5)$ appear in the second intersection square. □

Now let H_1' and H_1'' be 9-hyperplanes. Let $K \cap H_1'$ and $K \cap H_1''$ be extended to (11,5)-sets by the points P', Q' and P'', Q'', respectively. Set $\pi = H_0 \cap H_1' \cap H_1''$. Consider a projection φ from the plane π. Assume $|\pi \cap \{P', Q'\}| = 2$, then Table 1 and Remark 2.2 give that the type of $\varphi(H_1')$ is $(3,3,3,0)$. Hence π determines the intersection square $\begin{matrix} 3 & 3 & 3 \\ 6 & 6 & 6 \\ 6 & 6 & 6 \end{matrix}$, and the only possibility for H_1'' is a 15- or 18-hyperplane; a contradiction.

Hence $|\pi \cap \{P', Q'\}| = |\pi \cap \{P'', Q''\}| = 0$ or 1. For, if $|\pi \cap \{P', Q'\}| = 1$, then there is a $1-4-4$-parallel solid triple in H_1' (Table 1 (F) and Remark 2.2). Then the fourth intersection square of Lemma 4.6 shows that also in H_1'', there must be a $1-4-4$-parallel solid triple. So also here, using Table 1 (F) and Remark 2.2, $|\pi \cap \{P'', Q''\}| = 1$. Let us assume that π contains the points P' and P'' and does not contain the points Q' and Q''. Our next goal is to prove that $P' = P''$.

By Table 1 and Remark 2.2, the types of $\varphi(H_1')$ and $\varphi(H_1'')$ are (4,4,1,0) and $|K \cap H_1' \cap H_1''| = 1$ since a 4-solid does not lie in two 9-hyperplanes (Lemma 4.6). Set $K \cap H_1' \cap H_1'' = \{R\}$. Moreover the other two hyperplanes through $H_1' \cap H_1''$ are 15-hyperplanes (Lemma 4.6).

Assume that $P' \neq P''$ and consider another projection φ_ε from a plane ε in $H_1' \cap H_1''$ which contains P' and does not contain P'' or R.

We show that the type of $L_1 = \varphi_\varepsilon(H_1')$ is (4,3,1,1). Consider the $(11,5)$-set $(K \cap H_1') \cup \{P', Q'\}$ in H_1' which is the extension of the $(9,5)$-set $K \cap H_1'$. Every solid in H_1' through ϵ

intersects this $(11, 5)$-set in 5 or 2 points (Section 2). Going from the $(11, 5)$-set in H'_1 to the $(9, 5)$-set $K \cap H'_1$, we cancel the point P' which lies in ϵ. It is impossible that we have a 0-entry in the type of L_1, since a (9,5)-set in $PG(4, 3)$ has exactly one 0-solid $H_0 \cap H'_1$ and $\epsilon \not\subseteq H_0 \cap H'_1$. Hence, a 2-intersection of the $(11, 5)$-set becomes a 1-entry for the type of L_1; and a 5-intersection of the $(11, 5)$-set becomes a 4- or a 3-entry for the type of L_1. Now, the only possibility for the type of L_1 is $(4, 3, 1, 1)$ since the total of the 4 numbers must be 9.

We now show that the type of $L_2 = \varphi_\varepsilon(H''_1)$ is (5,2,1,1) or (4,2,2,1). Consider the $(11, 5)$-set $(K \cap H''_1) \cup \{P'', Q''\}$ in H''_1 which is the extension of the $(9, 5)$-set $K \cap H''_1$. Now ϵ does not contain a point of the $(11, 5)$-set in H''_1. Note that $\langle \epsilon, P'' \rangle$, which is the solid $H'_1 \cap H''_1$, does not contain Q'', since $Q'' \notin \pi$. Hence the two solids $\langle \epsilon, P'' \rangle$ and $\langle \epsilon, Q'' \rangle$ are different, and when we project $H''_1 \cap K$ from ϵ onto L_2, two entries of the type of L_2 differ a unit from the number of points of the (11,5)-set in H''_1 in the corresponding solids through ϵ in H''_1. As in the preceding paragraph, there is no 0-solid through ϵ in H''_1, so we need to decrease two different entries of the $(5, 2, 2, 2)$-type corresponding to the $(11, 5)$-set by one, giving $(5, 2, 1, 1)$ or $(4, 2, 2, 1)$.

Case 1. Construct $PG(2, 3)$ which represents the quotient geometry of ϵ. First suppose we have a (4,3,1,1)- and a (4,2,2,1)-line. We can fix the entries of the type of L_1 and L_2 without losing generality.

Namely, for the points on L_1, this is certainly true. Then we can use an elation with center $L_1 \cap L_2$ and axis L_1 to choose the weight of a point y on $L_2 \setminus L_1$. Next use the involutory perspectivity with axis L_1 and center y to choose the weights of the other points on L_2.

Since we projected from a plane ϵ which is skew to the 45-cap, all lines must sum to 0 (mod 3), because hyperplanes intersect K in 9, 15 or 18 points.

Consider the following picture of $PG(2, 3)$ where we number the points from 1 to 13.

10	11	12	13
	1	2	3
	4	5	6
	7	8	9

where $PG(2, 3)$ is considered to be the union of the affine plane of the points represented by the 3×3-grid of points $1, \ldots, 9$ and the line at infinity $10, \ldots, 13$, with 10 the point at infinity of the vertical lines of the 3×3-grid, 12 the point at infinity of the horizontal lines of the grid, 13 the point at infinity of the affine lines $\{1, 5, 9\}, \{2, 6, 7\}, \{3, 4, 8\}$, and 11 the point at infinity of the affine lines $\{3, 5, 7\}, \{1, 6, 8\}, \{2, 4, 9\}$.

Completing the picture of $PG(2, 3)$ and calculating the weights of the points modulo 3, we obtain a table of the following type where the $(4, 3, 1, 1)$-line L_1 is the line at infinity and where the points $12, 1, 2, 3$ of the description above form the line L_2, and where $a \in GF(3)$.

0	1	1	1
	2	2	1
	a	a	$a-1$
	$1-a$	$1-a$	$-a$

If we now fill in the explicit possibilities for the weights of the points of $PG(2,3)$, taking into account that every line must have a total weight of 9, 15 or 18; only a limited number of possibilities occur. If one considers such a possibility, one finds that there is a (3,3,2,1)-line L_3.

This line defines a 9-hyperplane intersecting the 45-cap in a $(9,5)$-set. This is always uniquely extendable to an (11,5)-set intersecting every solid in 2 or 5 points. Since the line is a (3,3,2,1)-line, necessarily, the plane ϵ must contain the two points which extend the (9,5)-set to the (11,5)-set; but then the projection from ϵ would imply that the line L_3 is a (3,3,3,0)-line since we lose two points in a 5-solid and in a 2-solid to the $(11,5)$-set.

So we get a contradiction.

Case 2. Now, suppose we have a (4,3,1,1)- and a (5,2,1,1)-line. Using the same arguments, we obtain a contradiction. □

Hence, the following lemma is valid.

Lemma 4.8 *For every* 9*-hyperplane* H*, we have* $|L \cap H| = 2$.

Denote by δ_i, $i = 1, \ldots, 55$, the 55 solids in H_0 that are contained in 9-hyperplanes. We have $|L \cap \delta_i| = 2$ by Lemma 4.8 and $|L \cap \delta_i \cap \delta_j| = 0$ or 1 when $i \neq j$; see the discussion following the proof of Lemma 4.7. Let $L \cap \delta_1 = \{P, Q\}$. There exist nine planes π_i, $i = 1, \ldots, 9$, in δ_1 that contain P and do not contain Q. If we project from π_i; a 9-hyperplane through π_i is projected onto a $(4,4,1,0)$-line (Table 1 (F) and Remark 2.2). From the fourth intersection square of Lemma 4.6, π_i lies in a second 9-hyperplane; so π_i lies in a second solid δ_{i+1}. Consequently, each point of L is on 10 of the solids δ_i, $i = 1, \ldots, 55$. Counting in two ways the number of flags (P, δ), where $P \in L$ and $P \in \delta$ with $\delta \in \{\delta_1, \ldots, \delta_{55}\}$, we get $10 \cdot |L| = 2 \cdot 55$. Therefore $|L| = 11$.

Lemma 4.9 *The set* L *is an* $(11,5)$*-set in* H_0.

Proof: All multiplicities in this proof are meant with respect to the 11-set L defined on the points of H_0.

Consider an empty plane π, with respect to L, and assume it lies in a 9-hyperplane of K. There is a one-to-one correspondence between the pairs of L and the fifty-five 2-solids to L which are the solids at infinity of the 9-hyperplanes of K. It follows from Lemma 4.6 that such an empty plane π is contained in two further 2-solids. For, the type of the projection from π of the 9-hyperplane is $(5,2,2,0)$ (Table 1 (E) and Remark 2.2), so it determines the 3×3 intersection square only containing the numbers 2, 5 and 8, and this intersection square has three parallel classes containing 9-solids.

Assume that δ is a w-solid with $2 < w \leq 9$; so there are at least two points of L not in δ. This w-solid is not contained in a 9-hyperplane with respect to K (Lemma 4.8). Fifty-five 2-solids are in one-to-one correspondence with the pairs of L. Hence such a 2-solid containing two points from $L \setminus \delta$ intersects δ in a 0-plane π. By the preceding paragraph, π is contained in three 2-solids and one w-solid which is forced to be a 5-solid.

To complete the proof, it remains to be checked that there cannot be 10- or 11-solids with respect to L. Assume there exists a 10- or an 11-solid S. No three of the points of

$L \cap S$ can be collinear, since there is a bijection between the fifty-five 2-solids and the pairs of L. Because there are at most 10 points on a cap in a solid, this shows that we cannot have 11-solids. Hence S is a 10-solid, and $S \cap L$ is an elliptic quadric Q. Every pair of the 10-solid S is contained in a 2-solid, which necessarily intersects S in a plane. This plane shares already 2 points with Q, so shares at least 4 points with Q. But this plane is contained in a 2-solid; a contradiction. □

Theorem 4.10 *The set* $K \cup L$ *is a* $(56, 20)$*-set.*

Proof: Each solid in H_0 contained in two 18- and one 9-hyperplane contains 2 points from L (Lemma 4.8) and each solid in H_0 contained in three 15-hyperplanes contains 5 points from L (Lemma 4.9). □

The 56-cap of Hill is the only $(56, 20)$-set in $PG(5, 3)$ [10]. The 11-hyperplanes of the 56-cap are the tangent hyperplanes to the elliptic quadric containing this 56-cap, with the tangent point belonging to the 56-cap. Since the group stabilizing the 56-cap acts transitively on the points of the 56-cap [7]; all these 11-hyperplanes are projectively equivalent; hence, the corresponding 45-caps are unique.

This finishes the discussion of this case.

4.2 Suppose there are 9-solids

Embed $AG(5, 3)$ in $PG(5, 3)$ by adding the hyperplane H_0 at infinity. Then H_0 is a hyperplane skew to this 45-cap in $PG(5, 3)$. We identify the affine points with the corresponding projective points.

By Lemma 4.5, we have two parallel 9-solids S_1 and S_2, lying in a hyperplane $H \equiv PG(4, 3)$. By Lemma 2.1, a 9-cap in $AG(3, 3)$ is always obtained by deleting a 1-plane of an elliptic quadric in $PG(3, 3)$. Hence, working in the projective space, $S_i \cap K$ is an elliptic quadric Q_i minus a point p_i, $i = 1, 2$. And p_1 and p_2 have the same tangent plane, lying in H_0, to respectively Q_1 and Q_2.

Suppose there is another 9-solid contained in H. Then, this solid contains at least 5 points of one of the two elliptic quadrics, so contains the elliptic quadric completely.

Denote by n_i the number of i-solids contained in H. Then we just showed that $n_9 = 2$.

We now will use parallel classes of solids in $H \setminus H_0$. A parallel class of solids in $H \setminus H_0$ consists of three solids of $H \setminus H_0$ intersecting in a fixed plane of $H \cap H_0$. Every parallel class of solids of $H \setminus H_0$ comes from an intersection square of Lemma 4.6. We count how many intersection squares of every type there are. The intersection squares of Lemma 4.6 differ from each other in the number of parallel classes of 15-hyperplanes they contain. Note that the latter intersection square of Lemma 4.6 only containing the number 5 cannot determine a parallel class in H since the three parallel solids in H would only contain 15 points in total, instead of the 18 points of $K \cap H$. Letting the plane π which determines the intersection square (see Remark 4.4) vary in the solid at infinity $H \cap H_0$; we denote by a_i the number of intersection squares with i parallel classes of 15-solids ($i = 0, \ldots, 3$);

hence a_0, a_1, a_2, respectively a_3, denote the number of intersection squares of the first, second, fourth, respectively third, type as in Lemma 4.6.

We have

$$\begin{aligned} a_0 + a_1 + a_2 + a_3 &= 40 \qquad (5)\\ a_1 + 2a_2 + 3a_3 &= 66 \qquad (6)\end{aligned}$$

where the first number equals the number of planes in the solid at infinity of H_0, and where the second number is equal to 66; the total number of parallel classes of 15-solids (Lemma 4.2). Let b_1 be the number of parallel $2-8-8$-solid triples in H and let b_2 be the number of parallel $5-5-8$-solid triples in H. Then

$$b_1 + b_2 = a_1 \qquad (7)$$

since these two types of solid triples only occur in intersection squares of the second type in Lemma 4.6.

We now express the spectrum of the 18-cap in H in terms of a_i and b_i: $n_0 = 2$ since we have one 0-solid at infinity and one 0-solid corresponding to the type $(9, 0, 9)$. Also $n_1 = 0$ since only the fourth intersection square of Lemma 4.6 contains a 1-solid. And in this intersection square, a 1-solid only lies in 9- and 15-hyperplanes, but this contradicts the fact that H contains 18 points of the 45-cap. Similarly, $n_3 = 0$ since a 3-solid only lies in the first and third type of intersection squares of Lemma 4.6. Only in the first type of intersection square, a 3-solid lies in a 18-hyperplane, but then the parallel class determined by the 3-solid would give rise to a 9-solid different from S_1 and S_2. This was excluded in the beginning of this section. And $n_2 = b_1$, since the only way of having a 2-entry in the type of an 18-hyperplane is $(2, 8, 8)$, which occurs b_1 times; $n_4 = a_2$ since a 4-solid only lies in the fourth square of Lemma 4.6 and this determines a $(7, 7, 4)$-type in H; $n_5 = 2b_2$ since a 5-solid, contained in H, lies only in the second intersection square of Lemma 4.6 and such a square intersects H in a (5,5,8)-triple containing two 5-solids; $n_6 = 3a_3+3(a_0-1)$, since the third intersection square yields three 6-solids in H and there is one intersection square of the first type, which determines the $(9, 0, 9)$-type, the other intersection squares of the first type yield three 6-solids in H; $n_7 = 2a_2$ since a 7-solid lies only in the fourth intersection square of Lemma 4.6, and such a square determines the $(7, 7, 4)$-type in H; $n_8 = 2b_1 + b_2$ since there are b_1 $(2, 8, 8)$-triples and b_2 $(5, 5, 8)$-triples giving respectively two and one 8-solids; $n_9 = 2$.

Applying (1) to $H \cap K$, we have

$$\begin{aligned} \sum i(i-1)n_i &= 18 \times 17 \times 13 \qquad (8)\\ \sum i(i-1)(i-2)n_i &= 18 \times 17 \times 16 \times 4. \qquad (9)\end{aligned}$$

Now $(8) - (9)/12 - 57 \times (5) - (6) - 58 \times (7)$ shows that $a_0 = 0$; while it should be at least one.

We have shown the following lemma.

Lemma 4.11 *There is no* 45*-cap in* $AG(5, 3)$ *having a* 9*-solid.*

We have discussed all possible configurations that can occur in a 45-cap. Only the 45-cap arising from deleting an 11-hyperplane from a 56-cap in $PG(5, 3)$ remains. This proves Theorem 1.2. □

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