41 is the largest size of a cap in $PG(4, 4)$

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July 15, 2002

Abstract

We settle the question of the maximal size of caps in $PG(4, 4)$, with the help of a computer program.

1 Introduction

A **cap** in $PG(k-1, q)$ is a set of points no three of which are collinear. If we write the n points as columns of a matrix we obtain a (k, n)-matrix such that every set of three columns is linearly independent, hence the generator matrix of a linear orthogonal array of strength 3. This is a check matrix of a linear code with minimum distance ≥ 4. We arrive at the following:

Theorem 1 *The following are equivalent:*

- *A set of n points in $PG(k-1,q)$, which form a cap.*
- *A q-ary linear orthogonal array of length n, dimension k and strength 3.*
- *A q-ary linear code $[n,n-k,4]_q$.*

Denote by $m_2(k,q)$ the maximum cardinality of a cap in $PG(k,q)$. Assume $q>2$. It is known that

$$m_2(2,q) = \begin{cases} q+1 & \text{if } q \text{ is odd} \\ q+2 & \text{if } q \text{ is even} \end{cases}$$

and $m_2(3,q)=q^2+1$. Only two values $m_2(k,q)$ are known when $q>2, k>3$. These are $m_2(4,3)=20$ (the Pellegrino caps [6]) and $m_2(5,3)=56$ (the Hill cap [5]). In this paper we are going to establish the following:

Theorem 2 $m_2(4,4)=41$.

The lower bound has been established by Tallini [7]in 1964. In the last section we will give two essentially different 41-caps in $PG(4,4)$. We have to prove that $PG(4,4)$ does not contain a 42-cap.
Assume there is a 42-cap $\mathcal{K} \subset PG(4,4)$. Denote by $a(i)$ the number of hyperplanes meeting $\mathcal{K}$ in precisely i points. Construct a quaternary $(5,42)$-matrix G with the points of the cap as columns. Put $\mathcal{K}=\{P_1,P_2,\dots,P_{42}\}$, where P_j corresponds to column j of G. Matrix G is a generator matrix of a quaternary linear code $\mathcal{C}$ of length 42 and dimension 5. Denote by A_i the number of code-words of weight i. The rows of G will be denoted by $v_i, i=1,2,3,4,5$. Let $0 \neq x=(x_1,x_2,\dots,x_{42}) \in \mathcal{C}$. Then $x=\sum_{i=1}^{5}\lambda_i v_i$, where $\lambda_i \in \mathbb{F}_4$. Consider the hyperplane $H=(\lambda_1,\dots,\lambda_5)^{\perp}$. We have $P_j \in H \iff x_j=0$. This shows that there is a 1-1 correspondence between hyperplanes intersection $\mathcal{K}$ in i points and 1-dimensional subspaces of code $\mathcal{C}$, whose nonzero vectors have weight $42-i$. This proves the following well-known fact:

Theorem 3 *Let $\mathcal{K} \subset PG(4,4)$ be a 42-cap and $\mathcal{C}$ a quaternary code generated by a matrix whose columns represent the points of $\mathcal{K}$. Denote by $a(i)$ the* *number of hyperplanes meeting $\mathcal{K}$ in precisely i points, by A_i the number of code-words of weight $i, i = 1, 2, \ldots, 42$. Then the following holds for all i :*

$$A_i = 3 \cdot a(42 - i).$$

It is known that the maximum possible minimum distance of a quaternary code of length 42 and dimension 5 is $d = 29$, see Brouwer's data base [3]. Theorem 3 shows that some hyperplane H must meet $\mathcal{K}$ in at least 13 points.

Lemma 1 *Let $\mathcal{K} \subset PG(4,4)$ be a 42-cap. There is a hyperplane H such that $\mid \mathcal{K} \cap H \mid \geq 13$.*

Clearly $\mathcal{K} \cap H$ is a cap in $PG(3,4)$. Its cardinality is therefore bounded by 17 from above. Our proof will consist of two steps: We will classify all caps in $PG(3,4)$ with at least 13 points, up to operation of the group $P\Gamma L(4,4)$. The second and decisive step is to run a program, which in each of these cases completes an exhaustive search for 42-caps intersecting a fixed hyperplane in a given cap of cardinality ≥ 13. The program is written in C++. The central recursive procedure is printed and explained in Section 3. The program needs about 1MB of memory. On a HP 712/60 workstation it runs from 17 hours when starting from the ovoid in $PG(3,4)$ up to 19 days starting from a 13-cap in $PG(3,4)$.

2 Caps in $PG(3,4)$

2.1 Caps in ovoids

We are going to review some basic facts of geometric algebra. For an introduction see Artin [1]. It is well-known that the maximum size of a cap in $PG(3,q), q > 2$ is $q^2 + 1$. Also, the only 17-cap in $PG(3,4)$ is the ovoid. Ovoids may be described as follows:
Let Q be a non-degenerate quadratic form defined on the vector space $V = V(2m, q)$. Denote by $(,)$ the symmetric bilinear form such that

$$Q(x + y) = Q(x) + Q(y) + (x, y)$$

for all $x, y \in \mathbb{F}_q$. Here we have specialized to the case of characteristic 2. Then (V, Q) is of one of two possible types, which are distinguished by the Witt index d, defined as the dimension of the largest totally isotropic subspace. $d = 2$ is called the (+)type, $d = 1$ the (-)type. The group of isomorphisms (the orthogonal group) is defined as the set of all elements in $GL(2m, q)$, which respect this quadratic form. It is denoted by $\Omega^+_{2m}(q)$ and $\Omega^-_{2m}(q)$, respectively. Here we are interested in the (-)type in dimension 4. The points of $PG(3, q)$ are the 1-dimensional subspaces of V. The collection of isotropic points form a cap $\mathcal{Q} \subset PG(3, q)$. It is easy to see that $\mathcal{Q}$ has $q^2 + 1$ points (see [1]). The order of $\Omega^-_4(q)$ (in characteristic 2) is

$$\mid \Omega^-_4(q) \mid = (q-1)(q^2+1)q^2(q+1)2 = 2(q^2-1)q^2(q^2+1).$$

It is known that $\Omega^-_4(q)$ is isomorphic to a subgroup of $P\Gamma L(2, q^2)$, in its action on the points of the projective line $PG(1, q^2)$. Put $G_0 = PGL(2, 16) = SL(2, 16)$. This is a simple group, which under this isomorphism maps to a subgroup of index 2 in $\Omega^-_4(4)$. As $P\Gamma L_2(q^2)/PGL_2(q^2)$ is cyclic it follows that the isomorphism carries $\Omega^-_4(4)$ to $G = SL_2(16) < \phi >$, where ϕ is induced by the field automorphism $x \mapsto x^4$. We study the operation of G on subsets of cardinality at least 13 of $PG(1, 16)$. As G_0 is 3-transitive there is one such orbit for each of the cardinalities 17,16,15,14. The operation on the 13-sets is similar to the operation on the complements, the 4-sets. The orders of our groups are $g_0 =\mid G_0 \mid = 17.16.15$ and $g = \mid G \mid = 2 \cdot g_0$. As $\binom{17}{4}$ does not divide g, there must be more than one orbit. For concrete calculations we use the representation of $\mathbb{F}_{16}$ as given in the last section. Consider the orbits of G_0 on 4-subsets. Because of 3-transitivity each such orbit has a representative $\{\infty, 0, 1, x\}$. The stabilizer of $\{\infty, 0, 1\}$ in G_0 is a symmetric group generated by the elements $\tau \mapsto \tau + 1$ and $\tau \mapsto 1/\tau$. The orbits of this group on 14 elements of $\mathbb{F}_{16} \setminus \mathbb{F}_2$ are the following:

$$\{\omega, \omega^2\}, \{\epsilon, \epsilon^3, \epsilon^4, \epsilon^{11}, \epsilon^{12}, \epsilon^{14}\} \text{ and } \{\epsilon^2, \epsilon^6, \epsilon^7, \epsilon^8, \epsilon^9, \epsilon^{13}\}.$$

It follows that G_0 has at most 3 orbits of 4-sets. The Frobenius automorphism ϕ fixes $\infty, 0$ and 1. As it maps $\epsilon \mapsto \epsilon^4$ it follows that the orbits of G on 4-sets agree with the orbits of G_0. In order to be on the safe side let us calculate the number of orbits. Here is the character-table of $SL_2(16)$, followed by the permutation character π_4 on the unordered 4-sets. The character-tables of the groups $PGL_2(q)$ have been given in [2].

The character-table of $SL(2,16)$							
	1	z	a^r	a^3	a^6	a^5	b^s
1	1	1	1	1	1	1	1
St	16	0	1	1	1	1	-1
χ_i	17	1	$\alpha^{ir}+\alpha^{-ir}$	$\alpha^{3i}+\alpha^{-3i}$	$\alpha^{6i}+\alpha^{-6i}$	$\alpha^{5i}+\alpha^{-5i}$	0
Θ_j	15	-1	0	0	0	0	$-(\beta^{js}+\beta^{-js})$
π_4	$\binom{17}{4}$	28	0	0	0	10	0

Here α, β are primitive 15^{th} and 17^{th} roots of unity, respectively. We have $i = 1, \ldots, 7; j = 1, \ldots, 8, r \in \{1, 2, 4, 7\}$. a and b are elements of orders 15 and 17 in $SL(2,16)$, respectively. Each nonidentity power of a has $< a >$ as centralizer, each nonidentity power of b has $< b >$ as centralizer. As we know the cycle type of each element of $SL(2,16)$ we can also determine the number of unordered 4-sets is fixes. These are the values of π_4. For example, a has type (15,1,1). Clearly $\pi_4(a) = 0$. As a^5 has type (3,3,3,3,3,1,1) we see that this element fixes precisely 10 unordered 4-sets, hence $\pi_4(a^5) = 10$.

The number of orbits of $SL(2,16)(= G_0)$ on unordered 4-sets is given by the scalar product $(\pi_4, 1)$, where 1 is the trivial character. We obtain

$$(\pi_4, 1) = \frac{1}{17.16.15}(\binom{17}{4} + 28.17.15 + 10.17.16) = 3.$$

We conclude that G_0 (and therefore also G) has three orbits of 4-sets. Denote these orbits by $\mathcal{O}_1, \mathcal{O}_2, \mathcal{O}_3$ where $\mathcal{O}_1$ is the shortest orbit. We have seen that every unordered triple is contained in exactly 2 members of $\mathcal{O}_1$, in 6 of $\mathcal{O}_2$ and in 6 of $\mathcal{O}_3$. By double counting we obtain $\mid \mathcal{O}_1 \mid = \binom{17}{3} \cdot 2/4 = 17.16.15/12 = 17.5.4 = 340$, and $\mid \mathcal{O}_2 \mid = \mid \mathcal{O}_3 \mid = 3 \mid \mathcal{O}_1 \mid$. It is reconforting to note that these numbers add up to $\binom{17}{4}$. The stabilizer of a representative of $\mathcal{O}_1$ therefore has order $g/340 = 24$ and the stabilizers of representatives of the remaining orbits have orders $24/3 = 8$.

Lemma 2 *G has three orbits of unordered 4-subsets in its action on the projective line. The corresponding stabilizers have orders 24,8 and 8,respectively. These orbits are also full orbits under G_0.*

So far we considered the action of $PGL_4(q)$ on quadratic forms. The group $\Omega_4^-(q)$ was defined as the stabilizer of on ovoid under this group. It is clear that the larger group $P\Gamma L_4(q)$ permutes quadratic forms. Denote the stabilizer of an ovoid under this group by $O_4^-(q)$. Let $q = 2^f$. Then $P\Gamma L_4(q)$ is an extension of $PGL_4(q)$ by the cyclic group of order f generated by the Frobenius mapping $x \mapsto x^2$. As the image of an ovoid under the Frobenius is an ovoid again, it follows that $O_4^-(q)$ is an extension of $\Omega_4^-(q)$ by a cyclic group of order f. It is in fact known that

$$O_4^-(q) \cong P\Gamma L_2(q^2)$$

and the operation of $O_4^-(q)$ on the points of the ovoid is similar to the action of $P\Gamma L_2(q^2)$ on the points of the projective line. Extending our earlier discussion of G on $PG(1,16)$ to the action of $P\Gamma L_2(16)$ we see that this group fuses the two long orbits of 4-sets under G into one orbit. This yields the following:

Lemma 3 *$P\Gamma L_2(16)$ has two orbits of unordered 4-subsets in its action on the projective line. The corresponding stabilizers have orders 48 and 8,respectively.*

Lemma 4 *Two different ovoids in $PG(3,4)$ intersect in less than 13 points.*

Proof: The quadratic form, which determines an ovoid, may be described by

$$Q(x_1, x_2, x_3, x_4) = x_1x_2 + x_3^2 + x_3x_4 + \omega x_4^2.$$

We start by exhibiting a set $\mathcal{N} = \{P_i =< p_i >| \ i = 1, 2, \ldots, 9\}$ of 9 points, which is contained in $V(Q)$ and in no other ovoid. We choose the p_i as follows:

i	p_i
1	(1,1,1,0)
2	(ω ,ω^2,1,0)
3	(ω^2 ,ω ,1,0)
4	(1,ω ,0,1)
5	(ω ,1,0,1)
6	(ω^2,ω^2,0,1)
7	(1,ω ,1,1)
8	(ω ,1,1,1)
9	(ω^2,ω^2 ,1,1)

Let $\rho \in \Omega_4^-(4)$ be described by

$$\rho(x) = (\omega x_1, \omega^2 x_2, x_3, x_4).$$

It is clear that ρ has order 3 and $\{P_1, P_2, P_3\}, \{P_4, P_5, P_6\}, \{P_7, P_8, P_9\}$ are orbits of ρ.
Assume $\mathcal{N} \subset V(Q')$, where $Q'(x_1, x_2, x_3, x_4) = \sum_{i=1}^4 \lambda_i x_i^2 + \sum_{i<j} \mu_{i,j} x_i x_j$. Consider the three equations given by $Q'(p_1) = Q'(p_2) = Q'(p_3) = 0$. The sum of these equations yields $\mu_{1,2} = \lambda_3$. Other linear combinations yield $\mu_{1,3} = \lambda_2$ and $\mu_{2,3} = \lambda_1$. In the same way the equations $Q'(p_4) = Q'(p_5) = Q'(p_6) = 0$ yield $\mu_{1,2} = \omega^2\lambda_4, \mu_{1,4} = \omega^2\lambda_2, \mu_{2,4} = \omega^2\lambda_1$. We can express all coefficients in terms of $\lambda_1, \lambda_2, \lambda_3$ and $\mu_{3,4}$. Finally, consider the equations $Q'(p_7) = Q'(p_8) = Q'(p_9) = 0$. The sum of these equations yields $\mu_{3,4} = \lambda_3$. Remain two independent vanishing linear combinations of λ_1 and λ_2. This shows $\lambda_1 = \lambda_2 = 0$. If $\lambda_3 = 0$, we obtain the contradiction $Q' = 0$. We can therefore normalize $\lambda_3 = 1$ and obtain $Q' = Q$.
We have shown that the only quadric containing $\mathcal{N}$ is $V(Q)$.
In order to complete the proof of the Lemma it suffices to show that each of the two orbits of 13-subsets of our ovoid under the action of the full orthogonal group contains a superset of $\mathcal{N}$. The union of $\mathcal{N}$ with a full orbit of ρ and the fixed point $(1:0:0:0)$ is a 13-cap, which is invariant under ρ. This is therefore a member of the short orbit of 13-caps under the action of the orthogonal group. Remains to show that not all 13-caps containing $\mathcal{N}$ belong to the short orbit. We can work in the group $PGL_2(16)$ in its action on the projective line. Elements of order 3 have precisely 2 fixed points. We can therefore change notation such that

$$\rho = (1, 2, 3)(4, 5, 6)(7, 8, 9)(10, 11, 12)(13, 14, 15)(X)(Y).$$

Here we have abbreviated P_i by i, hence $\mathcal{N} = \{1, 2, \ldots, 9\}$. Let $M = \mathcal{N} \cup \{10, 11, X, Y\}$. We claim that there is no element $\rho' \in PGL(2, 16)$ of order 3 stabilizing M. This will prove then that the corresponding 13-cap belongs to the long orbit under the full orthogonal group.
Assume ρ' is such element. ρ' operates on the complement $\{12, 13, 14, 15\}$ of M. It must have precisely one fixed point there. If this fixed point is 12, then ρ' agrees either with ρ or with ρ^{-1} in its action on $\{13, 14, 15\}$. Because of the sharp triple transitivity of $PGL_2(q)$ we conclude that $\rho' = \rho$ or $\rho' = \rho^{-1}$. This is a contradiction.

It follows that we can assume without restriction that 13 is a fixed point of ρ'. It follows that either $(12, 14, 15)$ or $(12, 15, 14)$ is a cycle of ρ'. In the former case, $(13, 15)$ is a cycle of $\rho\rho'$. It follows from the structure of PGL, that $\rho\rho'$ has order 2. As it maps $14 \mapsto 12$ we must have that $\rho' : 10 \mapsto 14$, contradiction. In the latter case $\rho\rho'^{-1}$ contains the cycle $(13, 15)$ and maps $: 14 \mapsto 12$. It must therefore map $12 \mapsto 14$. This forces $\rho' : 14 \mapsto 10$, another contradiction.■

It follows from Lemma 4 that each cap of size ≥ 13 in $PG(3, 4)$ is contained in at most one ovoid. This has the consequence that the automorphism group of such a cap, which is contained in an ovoid, equals the stabilizer of the cap under the action of the automorphism group of the ovoid. We arrive at the following:

Theorem 4 *We consider orbits of caps of size* ≥ 13 *contained in some ovoid in* $PG(3, 4)$ *under the action of* $P\Gamma L_4(4)$.

The following holds:

- *There is one such orbit for each of the cardinalities 17,16,15,14. The automorphism groups have orders* $17.16.15.4, 16.15.4, 120$ *and 24, respectively.*
- *There are two such orbits for cardinality 13. The automorphism groups have orders 48 and 8, respectively.*

The following is an ovoid:

The ovoid in $PG(3,4)$																
1	0	0	0	1	ω^2	1	ω^2	ω	1	ω^2	ω	ω^2	0	1	0	ω
0	1	0	0	1	ω	1	ω	1	ω	ω^2	0	1	ω^2	0	ω	ω^2
0	0	1	0	1	1	0	0	1	1	1	ω	ω	ω	ω^2	ω^2	ω^2
0	0	0	1	0	0	1	1	1	1	1	1	1	1	1	1	1

The first 13 columns yields a cap with 8 automorphisms. A cap with automorphism group of order 48 is obtained by restricting to columns $1, 2 \ldots, 12$ and 15.

2.2 Caps not contained in ovoids

Let us call a cap in $PG(3,4)$ **non-embeddable** if it is not contained in an ovoid. The maximal cardinality of a non-embeddable cap is 14. According to [4] there is exactly one $P\Gamma L(4,4)$-orbit of non-embeddable 14-caps. Here is a representative:

The complete 14-cap $\mathcal{K}_{14}$ in $PG(3,4)$													
1	0	0	0	1	ω^2	ω	1	ω^2	ω	1	0	ω	ω^2
0	1	0	0	1	ω	ω^2	1	ω	ω^2	0	1	ω	ω^2
0	0	1	0	1	1	1	0	0	0	1	1	1	1
0	0	0	1	0	0	0	1	1	1	1	1	1	1

Let G be the stabilizer of $\mathcal{K}_{14}$ in $P\Gamma L(4,4), G_0 = G \cap PGL(4,4)$. Then G_0 is a semidirect product of an elementary abelian group E_0 by $GL(3,2)$. We have $E_0 =< \alpha_1, \alpha_2, \alpha_3 >$, where

$$\alpha_1 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \alpha_2 = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \alpha_3 = \begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Points of $PG(3,4)$ are written as column vectors. G_0 is generated by E_0, τ and σ, where

$$\tau = \begin{pmatrix} 1 & 0 & 1 & \omega \\ 0 & 1 & 1 & \omega^2 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \sigma = \begin{pmatrix} 0 & 1 & 0 & \omega \\ 0 & 1 & 1 & \omega \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 1 \end{pmatrix}$$

Here τ has order 4,σ has order 7. The stabilizer of $\mathcal{K}_{14}$ in $P\Gamma L(4,4)$ is the direct product of G_0 and its center $< \alpha_3\phi >$ of order 2. ϕ denotes the Frobenius automorphism. In particular the automorphism group G is transitive on the points of $\mathcal{K}_{14}$.

It follows from [4] that there is precisely one orbit of complete 13-caps. A non-complete non-embeddable 13-cap must be embeddable in $\mathcal{K}_{14}$. As the automorphism group of $\mathcal{K}_{14}$ is transitive on its points we see that there is at most one orbit of non-embeddable non-complete 13-caps. It is easy to check that the 13-caps contained in $\mathcal{K}_{14}$ are indeed non-embeddable. We conclude that there are precisely two orbits of non-embeddable 13-caps. Here is a complete 13-cap:

The complete 13-cap $\mathcal{K}_{13}$ in $PG(3,4)$												
1	0	0	0	1	ω^2	ω	1	ω^2	ω	1	ω	0
0	1	0	0	1	ω	ω^2	1	ω	ω^2	0	1	ω^2
0	0	1	0	1	1	1	0	0	0	1	1	1
0	0	0	1	0	0	0	1	1	1	1	1	1

3 The main recursive procedure

3.1 The recursive procedure

We print here the heart of the C++ program, the recursive procedure. In the following subsection we will provide an explanation.

```
void rt(const int ti){
    int i,j,ii,z ;
    b1 a;
    if(ti>maxx){
        maxx=ti;
        if (ti>0)
            pri(maxx);
    }
    for(i=0;i<an[ti];i++){
        z=-1;
        for(j=i+1;j<an[ti];j++){
            a=tv[ti][j].b;
            if (!(bb[ti][i][a.i]&a.x)){
                z++;
                id[ti][z]=j;
            }
        }
        if (z+ti>=agk-2-lae){
            an[ti+1]=z+1;
            erg[ti]=tv[ti][i].n;
            for (j=0;j<=z;j++){
                tv[ti+1][j]=tv[ti][id[ti][j]];
                for (ii=0;ii<abl;ii++)
```

```
                        bb[ti+1][j][ii]=bb[ti][i][ii]|bb[ti][id[ti][j]][ii];
                    for (ii=0;ii<tai[tv[ti][i].n][tv[ti+1][j].n];ii++){
                        a=tab[tv[ti][i].n][tv[ti+1][j].n][ii];
                        bb[ti+1][j][a.i]|=a.x;
                    }
                }
                rt(ti+1);
            }
        }
    };
```

3.2 Description of the recursive procedure

We use homogeneous coordinates. A point in $PG(4,4)$ is therefore represented as $(x_0 : x_1 : x_2 : x_3 : x_4)$. Consider the hyperplane $H = (x_4 = 0)$. A cap $\mathcal{C} \subset H$ is given. Put $m =\mid \mathcal{C} \mid$. We wish to determine the 42-caps $\mathcal{K} \subset PG(4,4)$ satisfying $\mathcal{K} \cap H = \mathcal{C}$. As the pointwise stabilizer of H in $PGL_4(4)$ is transitive on the affine space $PG(4,4) \setminus H$, we can assume that point $F = (0:0:0:0:1) \in \mathcal{K}$. The program performs an exhaustive search for such caps, which contain F and intersect H precisely in $\mathcal{C}$.
The parameter ti describes the **depth** of the program. When the recursive procedure is called for the first time we have $ti = 0$. Whenever the recursion procedure is called with the new value of ti, we are given a cap $\mathcal{P}_{ti-1} \supset \mathcal{C} \cup \{F\}$ of size $m+1+ti$. Put $P_{-1} = \mathcal{C} \cup \{F\}$. For any cap $\mathcal{U} \subset PG(4,4)$ denote by $G(\mathcal{U})$ (the **good points**) the set of affine points $p \notin \mathcal{U}$, which complement $\mathcal{U}$ to a cap. The cardinality of $G(\mathcal{P}_{ti-1})$ is stored in $an[ti]$, the points of $G(\mathcal{P}_{ti-1})$ are stored in $tv[ti][i]$, where $i = 0\ldots, an[ti]-1$.
Table $bb[ti][p]$ contains the set $G(\mathcal{P}_{ti-1} \cup \{p\})$ for all $p \in G(\mathcal{P}_{ti-1})$. Another table $tab[p][q]$ stores the points on the line through points p and q. Naturally this table will contain only the information that is really needed in the program. With these preparations we are ready to describe the recursive procedure:

- If the depth reached is bigger than the current maximum, then the maximum is updated and some output is produced.

- The program runs then through all $p \in G(\mathcal{P}_{ti-1})$. Assume in the sequel p is given.
- The point p is used to extend the cap provided

$$\mid G(\mathcal{P}_{ti-1} \cup \{p\}) \mid + \mid \mathcal{P}_{ti-1} \mid \geq 41.$$

  Assume point p satisfies the last condition. Put

$$\mathcal{P}_{ti} = \mathcal{P}_{ti-1} \cup \{p\}.$$

The following steps are then performed by the program:

- Some parameters are updated.
- For all $q \in G(\mathcal{P}_{ti})$ the sets $G(\mathcal{P}_{ti} \cup \{q\})$ are determined and stored in $bb[ti+1][q]$. This is done using

$$G(\mathcal{P}_{ti} \cup \{q\}) = (G(\mathcal{P}_{ti-1} \cup \{p\}) \cap G(\mathcal{P}_{ti-1} \cup \{q\})) \setminus tab[p][q].$$

- Finally the recursive procedure is called again at depth $ti + 1$.

4 Appendix

4.1 The field $\mathbb{F}_{16}$

We describe $\mathbb{F}_{16}$ as an extension $\mathbb{F}_4(\epsilon)$ of $\mathbb{F}_4 = \{0, 1, \omega, \omega^2\}$. Our irreducible polynomial is $f(X) = X^2 + X + \omega$. This leads to the relation $\epsilon^2 + \epsilon + \omega = 0$. In order to see that $f(X)$ has maximal exponent write the elements of $\mathbb{F}_{16}$ as $\alpha\epsilon + \beta$, where $\alpha, \beta \in \mathbb{F}_4$. It follows $\epsilon^3 = (\epsilon + \omega)\epsilon = \epsilon^2 + \epsilon\omega = \epsilon + \omega + \epsilon\omega = \epsilon\omega^2 + \omega$. Proceeding in the same way we get $\epsilon^4 = \epsilon + 1, \epsilon^5 = \epsilon^2 + \epsilon = \omega$. As ω has order 3, it is clear that ϵ has order 15, thus $f(X)$ has maximal exponent. The remaining powers of ϵ are obtained by observing $\epsilon^{5+i} = \omega\epsilon^i, \epsilon^{10+i} = \omega^2\epsilon^i$. The additive structure is already determined:

$1 + \epsilon + \epsilon^4 = 0$	$1 + \epsilon^2 + \epsilon^8 = 0$	$1 + \epsilon^3 + \epsilon^{14} = 0$
$1 + \epsilon^6 + \epsilon^{13} = 0$	$1 + \epsilon^7 + \epsilon^9 = 0$	$1 + \epsilon^{11} + \epsilon^{12} = 0$

4.2 41-caps in $PG(4, 4)$

The columns of the following matrix M_1 form a 41-cap.

10000213010223333122103103230321021023032
01000132101013221322010121332022301101303
00100303223220123321330101023302112102012
00010032111103331223101030223133210010212
00001130331132032231021013303320332120102

Clearly we have written $\mathbb{F}_4 = \{0, 1, 2, 3\}$, where $2 + 3 = 2 \cdot 3 = 1$. Matrix M_1 is the generator matrix of a quaternary code $[41, 5, 28]_4$. The weight distribution of this code is

$$A_{28} = 120, A_{29} = 360, A_{31} = 288, A_{32} = 135, A_{37} = 120.$$

The columns of the following matrix M_2 form another 41-cap in $PG(4, 4)$.

10000112213322333222333020022100311310012
01000100200210110110130300230321231311222
00100012002001101101103302003312213311222
00010110011100011111111111111111111101011
00001001111122222211133333300022222200113

The weight distribution of the code generated by M_2 is

$$A_{24} = 9, A_{26} = 12, A_{28} = 105, A_{30} = 660$$

$$A_{32} = 90, A_{34} = 36, A_{36} = 51, A_{38} = 60.$$

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